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A Theory of Internal Ballistics Based on a Pressure-Index Law of Burning for Propellants

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IX. *A Theory of Internal Ballistics Based on a Pressure-Index Law of Burning for Propellants.*

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(*Communicated by Sir* GEORGE HADCOCK, F.R.S.)

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Introduction.

It has been felt for some time past that an extension of the present internal ballistic theory is necessary to take into consideration the case of propellants which do not burn according to the simple law generally adopted for M.D. Cordite.

This law, which makes the rate of reduction of the smallest linear dimension of a piece of propellant proportional to the first power of the gas pressure, renders the mathematical treatment of the ballistic problem, both as regards the closed vessel and the gun, comparatively easy. The law of burning must, however, from the nature of the case, be a more complicated phenomenon than is thus pre-supposed, and in the present paper the problem has been investigated assuming a rate of burning proportional to some power, less than unity, of the gas pressure. It is not pretended that such an assumption leads to finality, but in view of the many attempts made, chiefly by continental writers, to consider internal ballistics on this basis, and also since experiment seems to suggest such a law for many propellants, it was thought worth while to present a connected account of an investigation into this subject.

This paper was completed some four years ago, but publication has been delayed for unavoidable reasons.

As internal ballistics is mostly unfamiliar to English readers, the main principles upon which the theory has been built up are briefly outlined in the following sections.

1. *The Problem of Internal Ballistics.*

The problem, stated briefly, consists of the calculation of the maximum pressure and the muzzle velocity realised by the burning in a gun of a charge of propellant of known size and shape, the internal dimensions of the gun and the weight of the projectile being supposed given.

The determination of these quantities requires the complete analytical solution which gives the full history of events in the gun. Thus the pressure and shot velocity can theoretically be determined as functions of the forward travel of the shot or of the fraction of the charge weight burnt, or of any other variable as may be desired, and further information of ballistic importance can be obtained—such as the distance travelled by the shot up to the instant when the propellant charge is completely burnt (*i.e.*, entirely converted into gas), the position of the shot at maximum pressure, etc., etc. The knowledge afforded by such a complete solution is of fundamental importance in the theory of gun design, as it leads to methods which enable the characteristics of a gun to be determined in consonance with the specific performance desired.

2. *References.*

The bibliography of the subject is fairly extensive, and, although mainly in French and German, a certain amount of literature exists in English.

The following, while by no means complete, is a list of the more important books and memoirs, which will be referred to subsequently by their attached numbers :—

(1) Sir ANDREW NOBLE, "Artillery and Explosives." This volume contains NOBLE's collected researches.

(2) CHARBONNIER, "Balistique Intérieure" (1908). A leading French text-book which gives an excellent account of the subject. There is a good bibliography at the end of the book.

(3) GOSSOT ET LIOUVILLE, "Traité des Effets des Explosifs" (1920). A three-volume work treating the subject with a different view-point from CHARBONNIER.

(4) CRANZ, "Lehrbuch der Innere Ballistik." This is Volume II of a complete work on ballistics. Very full references are given.

(5) INGALLS, "Interior Ballistics." An American text-book.

(6) LONGRIDGE, "Internal Ballistics." An old English book which contains some interesting accounts of SARRAU's work.

(7) "Mémorial des Poudres et Salpêtres." A French scientific journal devoted to Explosives and ballistic subjects. Contains numerous papers by VIEILLE.

(8) "Mémorial de l'Artillerie de la Marine." Similar to the above. Since the war has appeared as "Mémorial de l'Artillerie Française."

(9) MANSELL, "Law of Burning of Modified Cordite," 'Phil. Trans. Roy. Soc.,' A (1908).

(10) HADCOCK, "Internal Ballistics," 'Roy. Soc. Proc.,' A, vol. 94 (1918).

(11) PROUDMAN, "The Principles of Internal Ballistics," 'Roy. Soc. Proc.,' A, vol. 100 (1921).

(12) HENDERSON and HASSÉ, "A Contribution to the Thermodynamical Theory of Explosions," 'Roy. Soc. Proc.,' A, vol. 100 (1921).

(13) PETAVEL and LEES, "On the Variation of the Pressure Developed during the Explosion of Cordite in Closed Vessels," 'Roy. Soc. Proc.,' A, vol. 79 (1906–07).

(14) PETAVEL, "The Pressure of Explosions—Experiments on Solid and Gaseous Explosives," 'Phil. Trans. Roy. Soc.,' A, vol. 205 (1905).

(15) LOVE and PIDDUCK, "Lagrange's Ballistic Problem," 'Phil. Trans. Roy. Soc.,' A, vol. 222 (1921).

3. *The Laws of Burning of Colloidal Propellants.*

All modern propellants are colloidal in nature, and are manufactured in definite shapes and sizes.

Thus cordite, as used in the British service, consists of long sticks in the form of right circular cylinders of small diameter.

Other shapes employed are: *tubular cordite*, viz., long hollow circular cylinders; *flake*, which consists of small square discs; *strip*, consisting of sheets of thickness, small compared with length and breadth, etc., etc.

On ignition, such propellants are converted into gas by regular burning, as distinct from the practically instantaneous conversion known as detonation, and this burning, either in a closed vessel or in a gun, is assumed to conform to the following two laws:—

(1) The burning takes place by parallel layers, *i.e.*, the rate of burning down the normal to the surface of each piece of propellant is the same, at any instant, at all points of the surface.

This is generally known as PIOBERT's Law, and is in conformity with experience.

(2) The rate of burning varies as some power of the pressure of the surrounding gases.

Analytically, if e is the thickness burnt down the normal at time t, then the rate of burning de/dt is given by $de/dt = k \cdot p^{\alpha}$, p being the pressure and k, α, positive constants. These constants will necessarily vary with the nature (chemical and physical) of the propellant, and much difference of opinion exists as to their value, particularly in the case of the index α.

Thus PIOBERT assumed $\alpha = 0$, SARRAU took $\alpha = \frac{1}{2}$, whilst for modern French smokeless propellants, CHARBONNIER (2) takes $\alpha = 1$. GOSSOT and LIOUVILLE (3), however, state that closed vessel experiments show conclusively that $\alpha = \frac{2}{3}$ for the propellants at present employed in the French service.

Most English writers (10, 12, 13) take $\alpha = 1$, although MANSELL (9) derived experimentally the formula $de/dt = a + bp$, where a, b are positive constants, a being small compared with b. This formula is also used by PROUDMAN (11). This law appears to be physically inadmissible, as it implies the burning of propellant under no pressure. Direct evidence on this point is available, as NOBLE ((1), pp. 523, 524) gives an account of an attempt of ABEL's to burn cordite in a vacuum.

The attempt failed, but NOBLE made no deduction from the negative result.

It must be realised that, in any case, law (2) is really empirical, as the rate of burning must depend on the state of the surrounding gases, so that the law should be of the form $de/dt = f\,(p, \mathrm{T})$, say, T being the absolute temperature.*

Little at present is known of temperature-time variations, either in the gun or in the closed vessel, and, in any case, the mathematical difficulties are sufficiently serious, even assuming a rate of burning dependent on pressure alone.

The assumption $\alpha = 1$ leads to comparatively simple analysis, but this linear law is probably insufficient to represent the phenomenon over a wide range of pressures, especially for propellants composed mainly of nitrocellulose, for which a value about $\alpha = \frac{1}{2}$ appears to be indicated experimentally.

The determination of the burning characteristics k, α of a propellant is made, in effect, by comparing theoretical and practical pressure-time rises obtained by burning propellants in a closed vessel† (*cf.* MANSELL (9), PETAVEL (14), for experimental details).

As has been mentioned above, such work has led to the assumption of a variety of values for α, even for the same propellant, and it seems advisable to work out the analytical consequences of the law $de/dt = kp^{\alpha}$, keeping k and α, for the present, as unknown constants.

4. *The Law of* NOBLE *and* ABEL.

When a mass of propellant is burnt in a closed vessel, the maximum pressure (p) is given by the formula

$$p = f\Delta/(1 - \eta\,\Delta), \qquad (1)$$

where f is a constant known as the "force" of the propellant, η is the "co-volume," taken by NOBLE and ABEL as equal to $1/\delta$, where δ is the density of the propellant, and Δ is the density of the gas produced: Δ is called, ballistically, the "density of loading." The formula (1), which is known as the pressure-density relation, was obtained by NOBLE and ABEL as the result of numerous closed vessel experiments: it bears an obvious resemblance to VAN DER WAAL'S equation of state.

The pressure at any instant whilst the propellant is still burning can be obtained from (1) by substituting for Δ the density of the gas produced up to that instant.

Thus, in C.G.S. units, if ϖ, V be the charge weight and capacity of the vessel respectively, we have $\Delta = \varpi/\mathrm{V}$, and, from (1), the maximum pressure produced when the charge is all converted into gas is $p = f\varpi/(\mathrm{V} - \eta\varpi)$.

At the instant when a fraction z of the charge has been burnt the instantaneous value of Δ, Δ_z, say, is given by $\Delta_z = \varpi z/(\mathrm{V} - \varpi\,(1 - z)/\delta)$, making allowance for the space occupied by the unburnt propellant. Thus, the pressure realised at that instant, p_z say, is, putting η for $1/\delta$,

$$p_z = f\varpi z/(\mathrm{V} - \eta\varpi) = zp. \qquad (2)$$

* The question of rates of burning is fully discussed, both theoretically and practically, by PROUDMAN (11).

† The theory of burning in the closed vessel is complicated by the cooling due to the walls of the vessel, the influence of the size of the propellant, the make-up of the charge, and the disposition in the vessel. The measurement of the gas pressure is also an operation of much difficulty.

5. *The Pressure in the Gun.*

In the case of a gun the volume in which the propellant is burning increases owing to the motion of the projectile, and the gases expand and do work. The formula (2) is, therefore, not applicable to this case. CHARBONNIER ((2), Chap. IV, p. 142) replaces (2) by the equation

$$p\,(c-\eta\varpi)+\tfrac{1}{2}(\gamma-1)\,\mu v^2=f\varpi z \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)^*$$

where c is the total capacity behind the shot at the instant when the fraction z of the charge weight is consumed, μ is the "effective" shot weight (to be explained later), v the shot velocity and γ an empirical constant. The formula (3) expresses CHARBONNIER's "Loi de détente," where $p\,(c-\eta\varpi)/(\gamma-1)$ represents the intrinsic energy in the gas, $\frac{1}{2}\mu v^2$ the effective kinetic energy of the shot, and $f\varpi z/(\gamma-1)$ the total energy derived by the combustion of the weight of charge ϖz.

Provided that the charge burnt is supposed all converted into gas before the projectile has begun to move, and assuming adiabatic expansion, (3) can be immediately derived with n, the expansion index, in place of γ. In the actual case the propellant is gradually converted into gas, and the above assumptions are not warranted, so that (3) must be regarded as partially empirical, the constant γ being adjusted so that calculated ballistic quantities agree with their experimentally determined values. CHARBONNIER takes the value $\gamma=1\cdot25$. HENDERSON and HASSÉ (12) deal with the question on thermodynamical grounds. They show that, at high pressures and temperatures, the following equation may be considered to hold

$$p\,(\mathrm{V}-\eta\varpi)=\mathrm{R}\varpi z\mathrm{T}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

V being the volume of the space behind the shot, T the absolute temperature, and R the gas constant.

As the shot moves, T falls, and the energy of the shot is derived from the loss of the heat content (h) of the gases; it is shown (Reference (12)) that h can be represented empirically with fair accuracy as a quadratic function of T. Up to the point of maximum pressure it is shown that the drop of temperature is not much greater than about 240°, which is less than 8 per cent. of the initial temperature.

Taking a mean constant value of T in (4) should, therefore, give a value for p less than $\pm$ 4 per cent. in error.

The derived formula

$$p=f'\varpi z/(\mathrm{V}-\eta\varpi), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

where f' is a constant, is the same as the closed vessel formula (2) with a different constant. It is a practice among certain ballisticians to use a formula of this type† with a constant

* This is known as the equation of RÉSAL, and was published by him in 1864.

† For example, *cf.* MATA, a Spanish writer on Ballistics. A French Translation of a paper by him is to be found in (8), Tome XXX, 1900.

adjusted to fit firing results, the idea being to link up "closed vessel" with "expanded" pressure.

The "expanded" pressure is obtained from the closed vessel pressure corresponding to the value of Δ at any instant by dividing by a constant, greater than unity, this being done to allow for the mean expansion since, of the various portions of gas given off during the burning, some will have expanded completely, and some not at all.

The reasoning, of course, is in no way rigorous, but the use of (5) leads to great simplification in the analysis (*vide* art. (20) *infra*).

In this paper the parallel treatments using both (3) and (5) will be given. It should be noted that PROUDMAN (11) derives a differential relationship between p, V and z for the case of varying capacity which is in accordance with the integral energy equation of CHARBONNIER.

6. *The Form Function.*

If a piece of propellant burns in accordance with PIOBERT'S law the surface S, when a fraction z of the volume has been consumed, can be expressed as $S = S_0 \phi(z)$, where S_0 is the original surface area. $\phi(z)$ is called by CHARBONNIER the form function, and depends only on the shape of the propellant.

The law of burning may now be written

$$dz/dt = A\phi(z)\, p^{\alpha} \qquad (6)$$

where $A = kS_0/V_0$, V_0 being the original volume.

It is usual among English writers to consider the fraction (f) of the *smallest* linear dimension (D) of the propellant grains remaining at any instant, and thus to express the law of burning in the form

$$D\frac{df}{dt} = -\beta p^{\alpha}, \qquad (7)^*$$

where β is now written for $k/2$.

When f becomes zero the propellant is completely converted into gas, or, as we shall say, is all "burnt."

D is called the "size" of the propellant, so that for long circular cords D is the diameter, for tube D is the width of the annulus, etc., etc.

The fraction z burnt at any instant can be expressed in terms of f, and if we write $z = \phi(f)$ (different from the ϕ of equation (6)), $\phi(f)$ is also called the form function.

It is easily shown that for all the shapes of propellant in use, viz., cord, tube, strip flake, etc.,

$$\phi(f) = (1-f)(1+\theta f), \qquad (8)$$

where $0 < \theta < 1$, the effect of the reduction in length of long sticks being neglected in comparison with the effect of the reduction in diameter, annulus, etc.

* No confusion need arise between the f of equation (1) and that of equation (7).

For circular cords $\theta = 1$, for tube $\theta = 0$, for flake θ is a small positive quantity.

Equation (6) can be written

$$\mathrm{D}\,\frac{dz}{dt} = \beta p^{\alpha}\sqrt{(1+\theta)^2 - 4\theta z} = \beta p^{\alpha}\psi(z), \text{ say}, \quad \ldots \ldots \quad (9)$$

and it is important to notice that, for $\theta = 0$, $\psi(z)$ is constant.

In such cases the propellant preserves a constant surface area during the burning, typical examples being tube and strip (neglecting the end effects). It is usual to express p in terms of the form function as follows: denoting the chamber capacity, cross sectional area of the bore, and forward travel of the shot by cap, σ and x respectively, we have $\mathrm{V} - \eta\varpi = \text{cap} - \eta\varpi + \sigma x = \sigma(x+l)$, say, where $\sigma l = \text{cap} - \eta\varpi$, so that (5) becomes

$$p = \frac{f'\varpi\phi(f)}{\sigma(x+l)}. \quad \ldots \ldots \quad (10)$$

In the case of a closed vessel we have, of course, $x = 0$.

7. *The Ballistic Problem—Mathematical Considerations.*

The problem naturally divides into two, the closed vessel problem and the gun problem.

Equations (7) and (10), with $x = 0$, are sufficient to determine the form of the pressure-time rise in a closed vessel, and the problem is soluble by quadratures for any value of α and any shape of propellant, *i.e.*, any value of θ. Full discussions will be found in the treatises of CHARBONNIER and of GOSSOT and LIOUVILLE.*

In the gun problem we have, in addition, the equation of motion of the shot, making three in all, two being simultaneous ordinary differential equations, and the problem can be reduced to the integration of a single ordinary equation of the second order. The treatment using the CHARBONNIER equations with $\alpha = 1$ will be found in his treatise. Using the simpler equation (10) instead of (3), and taking $\alpha = 1$, the integration presents no difficulties, as we have only to deal with a simple linear differential equation.

The case $\alpha \neq 1$ leads to a non-linear equation, the solution of which has been attempted by SARRAU and GOSSOT and LIOUVILLE, amongst others, a brief account of their work being given by CHARBONNIER ((2), Chap. IX), and a fuller account by GOSSOT and LIOUVILLE (3). The treatment is mainly approximate, and no direct integration by series, or otherwise, is attempted.

With given ballistic data it is useful to have a simple method of calculating the primary ballistic quantities, viz., maximum pressure (p_m) and muzzle velocity (V), and many attempts have been made to obtain monomial formulæ for p_m and V, usually from the results of extensive firing trials.

* See also PETAVEL and LEES (13).

It is shown in this paper that a monomial expression for p_m can be obtained from purely theoretical considerations for the case of a propellant of constant burning surface shape, whichever set of ballistic equations be employed (Art. 5).

In the case of the muzzle velocity, the calculation is complicated by the discontinuity in the analysis which occurs when the propellant is all burnt, and so it seems impossible to expect a monomial relation to exist.

The case of the maximum pressure is different, as this occurs, at the latest, at the instant at which the propellant is just completely burnt.

8. *Secondary Ballistic Factors.*

In actual practice, ballistic calculations have to take into account many disturbing factors of importance.

Consideration has to be given to the complications arising from driving-band resistance, frictional resistance down the bore, cooling effect of the walls of the gun, and so forth.

Various methods have been adopted to make allowance for band and frictional resistance, amongst others the modification of a certain parameter, and the assumption of a shot-start pressure (*cf.* CHARBONNIER (2)).

The correction for cooling is generally small in large calibre guns, and is not so important as for the closed vessel.

In this paper we shall consider only the ideal case of no band and no friction, and the treatment will be confined mainly to a mathematical discussion of the equations of Art. (9).

THE FUNDAMENTAL BALLISTIC EQUATIONS.

9. *The Two Sets of Equations.*

Equations (7) and (10), together with the equation of motion of the shot, give the set

$$p = \lambda \phi(f)/x, \qquad (11)$$

$$D\frac{df}{dt} = -\beta p^{\alpha}, \qquad (12)$$

$$\mu \frac{d^2x}{dt^2} = p, \qquad (13)$$

where λ, μ are constants depending on the gun dimensions and conditions of loading, x is written for $x + l$ occurring in (10), and the other quantities have been already defined.

The effects of the possible motion of the propellant charge, of the rotation of the shot, and of the recoil of the gun can be allowed for, approximately, by an addition to the mass of the shot, and are supposed to be taken up in the constant μ.

The second set of equations are those which are in general use by Continental ballisticians, and are as follows, the notation used being that of CHARBONNIER* ((2), p. 145):

$$p\,(c - \varpi') + \tfrac{1}{2}(\gamma - 1)\,\mu v^2 = f\varpi z, \qquad (14)$$

$$\frac{dz}{dt} = \mathrm{A}\phi\,(z)\,p^{\alpha}. \qquad (15)$$

$$\mu\,\frac{d^2x}{dt^2} = \sigma p. \qquad (16)$$

Here c is the volume behind the shot, σ the cross-section of the bore, $\varpi' = \eta\varpi$, and μ is the "effective" mass of the shot, differing from the μ of (13). No confusion need arise, as the two systems will be treated independently.

10. *The Motion after the Propellant is all Burnt.*

When the propellant is completely converted into gas it is assumed that expansion takes place according to the adiabatic law, pressure $\times$ (volume)^n = constant. The value of n cannot be assigned theoretically, and is best determined by making calculated and measured muzzle velocities to agree. The value $n = 1\cdot2$ is usually taken to apply to British propellants, but some writers—*e.g.*, MATA (footnote, p. 5) and LONGRIDGE (6)—treat expansion after burnt as isothermal.

The appropriate equations are thus, according to the first scheme,

$$\mu\,\frac{d^2x}{dt^2} = \mu v\,\frac{dv}{dx} = p \quad \text{and} \quad p\,.\,x^n = \text{constant},$$

v being the shot velocity.

If, therefore, x_b, v_b are values at burnt, *i.e.*, for $f = 0$, and x, v those at the muzzle,

$$v^2 - v_b^2 = \frac{2\lambda}{(n-1)\,\mu}\left[1 - \left(\frac{x_b}{x}\right)^{n-1}\right]. \qquad (17)$$

The muzzle pressure can also easily be calculated.

With isothermal expansion we have

$$v^2 - v_b^2 = \frac{2\lambda}{\mu}\log\frac{x}{x_b}, \qquad (18)$$

Similar formulæ arise, of course, using the CHARBONNIER equations. Thus provided, the equations of Art. 9 can be integrated, the scheme for ballistic calculation is complete.

* σ, μ are written here for CHARBONNIER's $\pi a^2/4$ and $\mu/10$ respectively.

11. *Reduction of the Equations of Scheme I.*

We now proceed to reduce the equations of each scheme in turn to a single differential equation, and shall refer to them as Scheme I (11 to 13), Scheme II (14 to 16).

Take a new variable F defined by

$$\frac{dF}{df} = -\frac{1}{\{\phi(f)\}^{\alpha}}. \qquad (19)$$

Eliminating p between (11) and (12) gives $dF/dt = \beta\lambda^{\alpha}/Dx^{\alpha}$, and eliminating p between (11) and (13)

$$x\frac{d^2x}{dt^2} = \frac{\lambda}{\mu}\psi(F), \qquad (20)$$

where

$$\psi(F) \equiv \phi(f).$$

Thus

$$\frac{dx}{dt} = \frac{\beta\lambda^{\alpha}}{Dx^{\alpha}}\frac{dx}{dF} = \frac{\beta\lambda^{\alpha}}{(1-\alpha)D}\frac{d}{dF}(x^{1-\alpha}),$$

so that

$$\frac{d^2x}{dt^2} = \frac{\beta^2\lambda^{2\alpha}}{(1-\alpha)D^2}\cdot\frac{1}{x^{\alpha}}\frac{d^2}{dF^2}(x^{1-\alpha}).$$

Writing

$$\left.\begin{aligned} y &= x^{1-\alpha} \\ k &= \frac{D^2(1-\alpha)}{\lambda^{2\alpha-1}\mu\beta^2} \end{aligned}\right\}, \qquad (21)$$

equation (20) becomes

$$y\frac{d^2y}{dF^2} = k\psi(F), \qquad (22)$$

the single equation required.

Theoretically, (22) gives the shot travel in terms of f, and so p in terms of f by (11): thus the maximum pressure can be found.

The shot velocity is determined in terms of f by

$$\frac{dx}{dt} = -\frac{\beta\lambda^{\alpha}}{(1-\alpha)D}\frac{dy}{dF},$$

so that the velocity at burnt can be found.

The travel to burnt comes from (22), and then the muzzle velocity is calculated from (17); therefore the solution of the ballistic problem is determined by the solution of (22).

The variables are p, v, x, f, t, and a variety of equations involving only two variables could be obtained: equation (22) appears to be the simplest.

12. *Reduction of the Equations of Scheme II.*

The presence of v^2 in (14) complicates the analysis considerably, and to simplify matters somewhat only the case $\phi(z) = 1$ will be considered, although the reduction can be carried out in general.

We take a new variable $\xi = c - \varpi'$, so that $d\xi = dc = \sigma\, dx$, and then (15) and (16) can be replaced by

$$\frac{dz}{d\xi} = \frac{A}{v\sigma} p^{\alpha}, \qquad (23)$$

$$\tfrac{1}{2}\mu \frac{d(v^2)}{d\xi} = p. \qquad (24)$$

Differentiating (14) and using (23) and (24)

$$\frac{d}{d\xi}(p\xi) + (\gamma - 1)\, p = \frac{f\varpi A}{\sigma} \frac{p^{\alpha}}{v},$$

i.e.,

$$v = \frac{f\varpi A}{\sigma} p^{\alpha}\xi^{\gamma-1} \Big/ \frac{d}{d\xi}(p\xi^{\gamma}). \qquad (25)$$

Eliminating v between (24) and (25) gives

$$p = \frac{\mu f^2 \varpi^2 A^2}{2\sigma^2} \frac{d}{d\xi}\left[\frac{p^{2\alpha}\xi^{2\gamma-2}}{\left\{\dfrac{d}{d\xi}(p\xi^{\gamma})\right\}^2}\right], \qquad (26)$$

which can be written

$$p = \frac{\mu f^2 \varpi^2 A^2 (1-\alpha)^2}{2\sigma^2} \frac{d}{d\xi}\left[\frac{\xi^{2\gamma(1-\alpha)-2}}{\left[\dfrac{d}{d\xi}\{(p\xi^{\gamma})^{1-\alpha}\}\right]^2}\right], \qquad (27)$$

Now change the variables ξ, p to η, q where

$$\left.\begin{aligned} \eta &= \xi^{\gamma(1-\alpha)} \\ q &= (p\xi^{\gamma})^{1-\alpha} \end{aligned}\right\}. \qquad (28)$$

Then (27) becomes, after reduction,

$$q^{\frac{1}{1-\alpha}} = \frac{\mu f^2 \varpi^2 A^2 (1-\alpha)}{\sigma^2 \gamma} \eta^{\frac{\gamma(2-\alpha)-1}{\gamma(1-\alpha)}} \frac{d^2\eta}{dq^2}. \qquad (29)$$

It is helpful to get rid of the troublesome powers of η and q which occur in (29), and this can be done by the further substitutions $q = Q^a$, $\eta = E^b$, and appropriate choice of a and b.

Carrying out the transformation and taking for a, b the values

$$a = \frac{1-\alpha}{3-2\alpha} \qquad b = \frac{2\gamma(1-\alpha)}{\gamma(3-2\alpha)-1},$$

(29) finally reduces to

$$QE\frac{d^2E}{dQ^2} - \frac{\gamma - 1}{\gamma(3-2\alpha)-1} Q\left(\frac{dE}{dQ}\right)^2 + \frac{2-\alpha}{3-2\alpha} E\frac{dE}{dQ} = \frac{\{\gamma(3-2\alpha)-1\}\sigma^2}{2(3-2\alpha)^2 \mu f^2 \varpi^2 A^2}. \qquad (30)$$

In terms of the original variables p, c, we have

$$\left.\begin{aligned} Q &= \{p\,(c-\varpi')^{\gamma}\}^{3-2\alpha} \\ E &= (c-\varpi')^{\frac{\gamma(3-2\alpha)-1}{2}} \end{aligned}\right\}. \quad (31)$$

Thus, theoretically, (30) determines the pressure in terms of the shot travel.

Integration of the Fundamental Equations.

This part of the paper will be divided into two sections, the first dealing with the case of propellants preserving a constant surface area during the burning, for which particular results of importance can be obtained (*vide* Art. 7), and the second with propellants of more general shape admitting a form function of the type

$$\phi(f) = (1-f)(1+\theta f).$$

Section I.

13. *The Fundamental Equation of Scheme I.*

For the propellants in question we have $\phi(f) = 1-f$, so that, from (19), assuming $\alpha < 1$, we have

$$F = \frac{\{\phi(f)\}^{1-\alpha}}{1-\alpha}, \quad \text{F being positive.}$$

Thus*

$$\psi(F) \equiv \phi(f) = \{F(1-\alpha)\}^{\frac{1}{1-\alpha}},$$

and putting

$$\left.\begin{aligned} X &= F(1-\alpha) \\ Y &= (1-\alpha)\,y/k^{\frac{1}{2}} \end{aligned}\right\}, \quad (32)$$

(22) becomes

$$Y\frac{d^2Y}{dX^2} = X^{\frac{1}{1-\alpha}}. \quad (33)$$

This equation is purely "numerical," *i.e.*, it determines Y as a function of X, whose form depends only on α, and is independent of all constants relating to the gun and charge.

Initially, since we are dealing with the ideal case (*vide* Art. 8), $\phi(f) = 0$, so that $F = 0$: also, by (21), $y = l^{1-\alpha}$ initially (Art. 9). Thus, for $t = 0$, $X = 0$, and $Y = (1-\alpha)\,l^{1-\alpha}/k^{\frac{1}{2}} = Y_0$, say.

Further, the shot velocity is zero initially, *i.e.*, $dx/dt = 0$ for $t = 0$, whence $dY/dt = 0$ for $t = 0$, since $Y = (1-\alpha)\,x^{1-\alpha}/k^{\frac{1}{2}}$ and x is finite initially.

* Throughout we shall assume in cases like this that the real positive root is taken.

But

$$\frac{dY}{dt} = \frac{dY}{dX}\cdot\frac{dX}{dF}\cdot\frac{dF}{df}\cdot\frac{df}{dt} = \frac{dY}{dX}(1-\alpha)\left(-\frac{1}{\phi^{\alpha}}\right)\left(-\frac{\beta}{D}p^{\alpha}\right),$$

i.e.,

$$\frac{dY}{dt} = \frac{\beta(1-\alpha)\lambda^{\alpha}}{Dx^{\alpha}}\frac{dY}{dX}, \qquad (34)$$

and so $dY/dX = 0$ for $t = 0$, *i.e.*, for $X = 0$.

The initial conditions appropriate to (33) are therefore

$$X = 0, \quad Y = Y_0, \quad \frac{dY}{dX} = 0. \qquad (35)$$

14. *The Series Solution of Equation* (33).

Since α is taken < 1 we have, by (33) and (35), $d^2Y/dX^2 = 0$, for $X = 0$, so we assume a solution of the form

$$Y = Y_0 + a_1X^{m_1} + a_2X^{m_2} + \dots, \qquad (36)$$

with $2 < m_1 < m_2 < \dots$ to satisfy the initial conditions.

The lowest power of X occurring in the product Yd^2Y/dX^2 is X^{m_1-2} and this must equal $X^{\frac{1}{1-\alpha}}$, otherwise it must vanish, so that $m_1(m_1-1) = 0$, which is impossible. Thus $m_1 = (3-2\alpha)/(1-\alpha)$ and the next lowest powers in the product are X^{2m_1-2} and X^{m_2-2}. The coefficients of these must vanish separately, or else their exponents must be equal. For the coefficient of X^{2m_1-2} to vanish we must have $a_1 = 0$, which is not possible.

Hence we take $m_2 - 2 = 2m_1 - 2$, *i.e.*, $m_2 = 2m_1$, and, for similar reasons,

$$m_3 - 2 = m_1 + m_2 - 2, \text{ or } m_3 = 3m_1,$$

and so on. The series (36) is therefore

$$Y = Y_0 + a_1X^{\frac{3-2\alpha}{1-\alpha}} + a_2X^{\frac{2(3-2\alpha)}{1-\alpha}} + a_3X^{\frac{3(3-2\alpha)}{1-\alpha}} + \dots. \qquad (37)$$

Determining the coefficients, we have

$$a_1 = \frac{(1-\alpha)^2}{(2-\alpha)(3-2\alpha)}\frac{1}{Y_0}; \quad a_2 = -\frac{(1-\alpha)^4}{(2-\alpha)(3-2\alpha)(5-3\alpha)(6-4\alpha)}\frac{1}{Y_0^3}, \text{ etc.,}$$

and it is easily seen that a_3 is a multiple of $1/Y_0^5$, a_4 of $1/Y_0^7$, and so on.

Thus (37) can be written

$$\frac{Y}{Y_0} = 1 + A_1\left(\frac{X^{\frac{3-2\alpha}{1-\alpha}}}{Y_0^2}\right) + A_2\left(\frac{X^{\frac{3-2\alpha}{1-\alpha}}}{Y_0^2}\right)^2 + A_3\left(\frac{X^{\frac{3-2\alpha}{1-\alpha}}}{Y_0^2}\right)^3 + \dots, \qquad (38)$$

where $A_1, A_2 \dots$, are numerical coefficients depending on α alone.

We now put

$$z = \frac{X^{\frac{3-2\alpha}{1-\alpha}}}{Y_0^2}. \qquad (39)$$

and write

$$Y = Y_0 f(z), \qquad (40)$$

so that $f(z)$ is a function of z, whose form is independent of all gun and charge constants.

If $1/(1-\alpha)$ is an integer, *i.e.*, if $\alpha = 0, \frac{1}{2}, \frac{2}{3} \dots$, the solution can be written down as a Maclaurin expansion, which, of course, agrees with (38).

15. *The Monomial Relation for Maximum Pressure.*

The pressure is given by (11), and, in terms of X and Y,

$$p = \lambda \left(\frac{1-\alpha}{k^{\frac{1}{2}}}\right)^{\frac{1}{1-\alpha}} \left(\frac{X}{Y}\right)^{\frac{1}{1-\alpha}}. \qquad (41)$$

The value of X at maximum pressure is given by

$$Y - X\frac{dY}{dX} = 0,$$

since Y is never zero.

Using (39) and (40) this equation becomes

$$f(z) = \frac{3-2\alpha}{1-\alpha} z f'(z), \qquad (42)$$

which is purely numerical, and its solution, z_m say, depends only on α. Thus, for a given propellant with a definite α, z_m can be determined once and for all.

Substituting for k, X, Y from (21), (39) and (40), in (41) we have, denoting the maximum pressure by p_m,

$$p_m = \frac{\{(1-\alpha)\, z_m\}^{\frac{1}{3-2\alpha}}}{\{f(z_m)\}^{\frac{1}{1-\alpha}}} \left(\frac{\lambda^2 \mu \beta^2}{D^2 l}\right)^{\frac{1}{3-2\alpha}}. \qquad (43)$$

This is the monomial expression referred to, and is a most important formula: we shall write it as

$$p_m = G(\alpha) \left(\frac{\lambda^2 \mu \beta^2}{D^2 l}\right)^{\frac{1}{3-2\alpha}}, \qquad (44)$$

so that G (α) is a function which can be tabulated in terms of α. We notice, from (10) and (11), that λ is proportional to ϖ/σ; also μ, equation (13), is proportional to w_1/σ, where w_1 is the "effective" shot weight. Thus, from (43), for a propellant of given burning characteristics, p_m varies as $(\varpi^2 w_1/\sigma^2 D^2 E)^{\frac{1}{3-2\alpha}}$, where $E = \sigma l$ is the initial air space in the gun chamber, *i.e.*, the chamber capacity less the volume occupied by the propellant charge.

16. *Some Remarks on the above Result.*

(*a*) When $\alpha = 1$ we obtain a result in conformity with (44), for then, from equations (11) to (13), we find

$$p = \frac{\lambda^2 \mu \beta^2}{D^2} \cdot \frac{\log \dfrac{x}{l}}{x},$$

so that p is a maximum when $\log x/l = 1$, *i.e.*, $x = le$ ($e = 2 \cdot 718 \ldots$).

Thus

$$p_m = \frac{1}{e} \cdot \frac{\lambda^2 \mu \beta^2}{D^2 l},$$

which agrees with (44) provided $G(1) = 1/e$.

(*b*) Keeping everything constant except the charge weight, the maximum pressure varies approximately as $(\varpi)^{\frac{2}{3-2\alpha}}$, since the variations in w_1/E with the charge weight are usually small in large calibre guns. Thus, for $\alpha = 1$, the pressure should vary as the square of the charge, and this is approximately the case for M.D. tubular cordite. For American nitro-cellulose propellants, which are manufactured in shapes approximately of constant burning surface, firings have shown that p_m varies as about $\varpi^{1 \cdot 4}$, which corresponds to an α of about $0 \cdot 8$. This figure is not very reliable, because it is probably derived from firings in small calibre guns, and it is mainly of interest as indicating a value of α definitely different from unity.

17. *Calculation of the Shot Travel to Burnt and the Muzzle Velocity.*

From (32) $X = \{\phi(f)\}^{1-\alpha}$, so that at burnt $X = 1$, and therefore $z = 1/Y_0^2$ from (39); thus the travel to burnt can be calculated from the equation $Y = Y_0 \mathcal{L}(z)$, since $Y = (1 - \alpha)\, x^{1-\alpha}/k^{\frac{1}{2}}$. The shot velocity is given by

$$v = \frac{dx}{dt} = \frac{dY}{dt} \Big/ \frac{dY}{dx} = \frac{\beta \lambda^\alpha k^{\frac{1}{2}}}{D(1-\alpha)} \frac{dY}{dX},$$

using (34), and, in terms of z, from (39) and (40),

$$v = \frac{3 - 2\alpha}{(1-\alpha)^2} \cdot \frac{\beta \lambda^\alpha k^{\frac{1}{2}}}{D} \cdot Y_0^{\frac{1}{3-2\alpha}} z^{\frac{2-\alpha}{3-2\alpha}} \mathcal{L}'(z).$$

Substituting for k from (21), we have

$$v = \frac{3 - 2\alpha}{(1-\alpha)^{3/2}} \cdot \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}} \cdot Y_0^{\frac{1}{3-2\alpha}} z^{\frac{2-\alpha}{3-2\alpha}} \mathcal{L}'(z), \quad \ldots \ldots \ldots \quad (45)$$

and the velocity at burnt is found by putting $z = 1/Y_0^2$ in the above, and then the muzzle velocity from (17).

18. *Ballistic Calculations.*

The calculation of ballistics for a constant burning surface propellant with a given pressure index α thus requires the tabulation of the functions $\mathcal{L}(z)$, $\mathcal{L}'(z)$ for $z = 0$ upwards, the solution z_m of (42), and the determination of the three parameters $\lambda^2\mu\beta^2/D^2 l$, λ/μ and $1/Y_0^2 = D^2/(1-\alpha)\, l^{2-2\alpha}\lambda^{2\alpha-1}\mu\beta^2$.

The evaluation of $\mathcal{L}(z)$ and $\mathcal{L}'(z)$ is best effected by a process of numerical integration, and will be discussed later, although the series solution suffices for the calculation with small values of z.

The tables must be comprehensive enough to include values corresponding to $z = 1/Y_0^2$, and a rough estimate can be made of this quantity for guns and charges normally employed (*vide* Art. 29).

There are two important points to be noticed about the analysis given above. Firstly, it may happen that the value $(1/Y_0^2)$ of z at burnt is less than z_m, in which case p_m, as given by (43), is the theoretical maximum pressure only, the true maximum being the pressure at burnt, since, after burnt, the pressure must necessarily diminish because of expansion of the gases.

Secondly, the charge may not be completely burnt inside the gun, and then the value of z at the muzzle is determined from $Y = Y_0\, \mathcal{L}(z)$, and the muzzle velocity by (45).

19. *The Effect of the Pressure Index on the Position of Burnt.*

In most cases, other things being equal, the travel of the shot up to the instant the propellant is all burnt increases as the charge weight decreases, but, for $\alpha = \frac{1}{2}$, an interesting difference is to be noticed. For then $1/Y_0^2 = 2D^2/\mu\beta^2 l$, *i.e.*, z at burnt is nearly independent of the charge weight, and therefore so is the shot travel to burnt, except in so far as the value of l is affected.

It follows that, for low densities of loading, charges of varying weights but of the same size should be burnt for roughly the same shot travel in the gun, provided $\alpha = \frac{1}{2}$. This is important for howitzers, which usually fire a graded series of charges, since it is necessary for good shooting to have the charge well burnt in the gun.

To illustrate numerically, we have for charges ϖ and 2ϖ, and $\alpha = 1/2$,

$$\left(\frac{1}{Y_0^2}\right)_{\varpi} \Big/ \left(\frac{1}{Y_0^2}\right)_{2\varpi} = \frac{(l)_{2\varpi}}{(l)_{\varpi}} = \frac{1 - \dfrac{2\Delta}{1\cdot 58}}{1 - \dfrac{\Delta}{1\cdot 58}},$$

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where Δ is the density of loading for the charge ϖ and $1\cdot58$ is the density of the propellant (the usual value). With $\alpha = 3/4$ we have

$$\left(\frac{1}{Y_0^2}\right)_{\varpi} \bigg/ \left(\frac{1}{Y_0^2}\right)_{2\varpi} = \frac{(l\lambda)_{2\varpi}}{(l\lambda)_{\varpi}} = \left[\frac{\left(1 - \dfrac{2\Delta}{1\cdot58}\right)^2}{1 - \dfrac{\Delta}{1\cdot58}}\right]^{1/2}$$

Taking $\Delta = 0\cdot1$, these ratios are respectively $0\cdot923$ and $1\cdot365$.

20. *The Monomial Relation for Maximum Pressure Deduced from the Equations of Scheme II.*

We now discuss the series solution of equation (30).

Assuming zero shot-start pressure, it is seen from equations (31) that, initially, $Q = 0$ and $E = E_0$, say, E_0 being > 0.

If we write $E - E_0 = a_1 Q^{m_1} + a_2 Q^{m_2} + \ldots$ where $0 < m_1 < m_2 < \ldots$, it is seen on substitution in the differential equation that $m_1 = 1$, and the constant a_1 is determined, so that the initial value of dE/dQ is assigned.

Equation (25) also shows that for v to vanish initially, $Q^{\frac{2-\alpha}{3-2\alpha}} dE/dQ$ must be zero for $Q = 0$, which is certainly true if $\alpha < 1$.

We now put (30) in the "numerical" form

$$QE' \frac{d^2E'}{dQ^2} - k_1 Q \left(\frac{dE'}{dQ}\right)^2 + k_2 E' \frac{dE'}{dQ} = 1, \quad \ldots \ldots \ldots \quad (46)$$

where k_1, k_2 are positive constants (since $\gamma > 1$, $\alpha < 1$), depending on γ and α only, and

$$E = \frac{\{\gamma(3 - 2\alpha) - 1\}^{1/2}\sigma}{(3 - 2\alpha)\sqrt{2\mu}\, f\varpi A} E' = \kappa E', \text{ say}, \quad \ldots \ldots \ldots \quad (47)$$

and assume a solution

$$E' = E_0' + A_1 Q + A_2 Q^2 + \ldots \quad \ldots \ldots \ldots \quad (48)$$

It is easily seen, on substituting in (46) and determining A_1, A_2, that A_1, A_2, A_3, ... are multiples of $1/E_0'$, $1/E_0'^3$, $1/E_0'^5$, ... respectively, so that we have

$$\frac{E'}{E_0'} = 1 + a_1\left(\frac{Q}{E_0'^2}\right) + a_2\left(\frac{Q}{E_0'^2}\right)^2 + a_3\left(\frac{Q}{E_0'^2}\right)^3 + \ldots, \quad \ldots \ldots \quad (49)$$

where a_1, a_2, a_3, ... are coefficients depending on γ and α. Thus, as in Art. (14), we can write

where

$$\left.\begin{aligned} E' &= E_0'\chi(\zeta) \\ \zeta &= \frac{Q}{E_0'^2} \end{aligned}\right\}, \quad \ldots \ldots \ldots \ldots \quad (50)$$

$\chi(\zeta)$ being a function independent of all gun and charge constants. From (31), $p = Q^{\frac{1}{3-2\alpha}}/(\kappa E')^{\frac{2\gamma}{\gamma(3-2\alpha)-1}}$, so that, at maximum pressure,

$$\frac{1}{3-2\alpha}\frac{1}{Q} - \frac{2\gamma}{\gamma(3-2\alpha)-1}\frac{1}{E'}\frac{dE'}{dQ} = 0,$$

which, using (50), leads to

$$\chi(\zeta) = \frac{2\gamma(3-2\alpha)}{\gamma(3-2\alpha)-1}\zeta\chi'(\zeta), \quad \ldots \ldots \ldots \ldots \quad (51)$$

a numerical equation of which the solution, ζ_m say, is invariant as regards gun and charge conditions.

From (31) we have $E_0 = (c' - \varpi')^{\frac{\gamma(3-2\alpha)-1}{2}}$, where c' is the chamber capacity of the gun, so that $c' - \varpi'$ is the initial air-space, and therefore, substituting for κ from (47), we find

$$p_m = \left\{\frac{(3-2\alpha)\sqrt{2}}{\sqrt{\gamma(3-2\alpha)-1}}\right\}^{\frac{2}{3-2\alpha}} \cdot \frac{\zeta_m^{\frac{1}{3-2\alpha}}}{\{\chi(\zeta_m)\}^{\frac{2\gamma}{\gamma(3-2\alpha)-1}}} \cdot \left\{\frac{\mu f^2 \varpi^2 A^2}{\sigma^2 (c' - \varpi')}\right\}^{\frac{1}{3-2\alpha}} \quad . \; . \; . \quad (52)$$

This formula, as regards the part involving gun and charge constants, is precisely the same as (44), since A is proportional to 1/D, so that both schemes of equations lead to the same result in this particular; but the multiplying constant in the present case depends on γ as well as on α. This is the fundamental difference between the two systems of ballistics: according to the first, allowance is made for the expansion of the gases while the propellant is still burning by a modification of the constants in the expression of the NOBLE–ABEL law: CHARBONNIER, on the other hand, attempts to construct an equation on thermodynamical principles which must necessarily involve the expansion index. It is of interest that the two systems lead to similar results, at any rate as regards the ballistic behaviour of propellants of constant burning surface shape. The result for $\alpha = 1$ can be deduced from (52) as a limiting case, for the analysis of Art. 12 is valid so long as $\alpha < 1$, however small $1 - \alpha$ may be, and so equation (46) with $\alpha = 1$ is the appropriate limiting form.

In terms of ζ and χ it is

$$\zeta\chi\frac{d^2\chi}{d\zeta^2} - \zeta\left(\frac{d\chi}{d\zeta}\right)^2 + \chi\frac{d\chi}{d\zeta} = 1.$$

This equation is of the homogeneous type, and can be integrated completely (equation (46) is, of course, also homogeneous, but the reduced equation cannot be integrated in finite terms), and we find the primitive

$$\chi(\zeta) = c\zeta^{\frac{1}{2}}\left(\zeta^{\kappa} + \frac{\zeta^{-\kappa}}{4\kappa^2 c^2}\right),$$

where c, κ are arbitrary constants.

The only solution satisfying the initial conditions $\zeta = 0$, $\chi(\zeta) = 1$ is clearly $\chi(\zeta) = 1 + \zeta$, and then (51) gives $\zeta_m = \dfrac{\gamma - 1}{\gamma + 1}$, $\chi(\zeta_m) = \dfrac{2\gamma}{\gamma + 1}$, so that, from (52),

$$p_m = \frac{1}{\gamma}\left(\frac{2\gamma}{\gamma + 1}\right)^{-\frac{\gamma+1}{\gamma-1}} \frac{\mu f^2 \varpi^2 A^2}{\sigma^2 (c' - \varpi')},$$

agreeing with CHARBONNIER's result (*vide* (2), p. 240).

The parameter $\mu f^2 \varpi^2 A^2/\sigma^2 (c' - \varpi')$ corresponds to $\lambda^2 \mu \beta^2/D^2 l$ of the first scheme of equations, their ratio being the ratio of the squares of the constants "f" defining the "force" of the propellant (Art. 4).

Comparing the above expression with that of Art. 16 (*a*) we have, denoting the constants "f" of Schemes I and II by f_1, f_2, respectively,

$$f_1^2/f_2^2 = \frac{e}{\gamma} \cdot \frac{1}{\left(1 + \frac{\gamma - 1}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}},$$

so that for $\gamma = 1 \cdot 25$, $f_1/f_2 = 1 \cdot 08935$. This illustrates the argument concerning "expanded" pressures (Art. 5).

SECTION II.

The analysis for propellants of other than constant burning surface shape is complicated, and, using the equations of Scheme II, it is almost impossible to deal with at all elegantly, so it is proposed to discuss here only the results arising from the equations of Scheme I.

21. *The Determination of the Function ψ (F) in Special Cases.*

Equation (19) can be re-written

$$\frac{dF}{d\phi} = \frac{1}{\phi^a \sqrt{(1 + \theta)^2 - 4\theta\phi}},$$

remembering that $d\phi/df$ is negative.

Since $F = 0$ for $\phi = 0$, we have

$$F = \int_0^\phi \frac{d\phi}{\phi^a \sqrt{(1 + \theta)^2 - 4\theta\phi}},$$

and, writing

$$\phi = \frac{(1 + \theta)^2}{4\theta} \sin^2 \chi,$$

this becomes

$$F = \frac{2}{1 + \theta} \left\{\frac{(1 + \theta)^2}{4\theta}\right\}^{1-a} \int_0^\chi \frac{d\chi}{\sin^{2a-1} \chi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

We proceed to consider one or two special cases.

(a) $\alpha = \frac{1}{2}$. We have $F = \chi \theta^{-\frac{1}{2}}$, so that

$$\psi(F) \equiv \phi(f) = \frac{(1+\theta)^2}{4\theta} \sin^2 (F\theta^{\frac{1}{2}}) \quad \ldots \quad (54)$$

Equation (22) becomes

$$Y \frac{d^2Y}{dX^2} = \sin^2 X, \quad \ldots \quad (55)$$

where

$$\left.\begin{aligned} X &= F\theta^{\frac{1}{2}} \\ Y &= \frac{\theta\beta\sqrt{8\mu}}{D(1+\theta)}\, y = \frac{\theta\beta\sqrt{8\mu}}{D(1+\theta)}\, x^{\frac{1}{2}} \end{aligned}\right\}. \quad \ldots \quad (56)$$

The initial conditions are easily shown to be

$$X = 0, \qquad Y = \frac{\theta\beta\sqrt{8\mu l}}{D(1+\theta)} = Y_0, \qquad \frac{dY}{dX} = 0. \quad \ldots \quad (57)$$

(b) $\alpha = \frac{3}{4}$. Here the integral in (53) is $I = \int_0^{\chi} d\chi/\sqrt{\sin\chi}$, which can be expressed in terms of Jacobian elliptic functions, as

$$\sin\chi = \mathrm{cn}^2\left(\frac{I}{\sqrt{2}}\right) \quad \left(\text{mod.}\ \frac{1}{\sqrt{2}}\right),$$

so that

$$\psi(F) = \frac{(1+\theta)^2}{4\theta}\, \mathrm{cn}^4\left(\frac{F}{2}\, \theta^{\frac{1}{4}} (1+\theta)^{\frac{1}{2}}\right) \quad \left(\text{mod.}\ \frac{1}{\sqrt{2}}\right).$$

Writing

$$\left.\begin{aligned} X &= \frac{F}{2}\, \theta^{\frac{1}{4}} (1+\theta)^{\frac{1}{2}} \\ Y &= \frac{2\theta^{\frac{3}{4}}\lambda^{\frac{1}{2}}\mu^{\frac{1}{2}}\beta}{D(1+\theta)^{\frac{1}{2}}}\, y = \frac{2\theta^{\frac{3}{4}}\lambda^{\frac{1}{2}}\mu^{\frac{1}{2}}\beta x^{\frac{1}{2}}}{D(1+\theta)^{\frac{1}{2}}} \end{aligned}\right\}, \quad \ldots \quad (58)$$

we get the numerical form

$$Y \frac{d^2Y}{dX^2} = \mathrm{cn}^4 X \quad \left(\text{mod.}\ \frac{1}{\sqrt{2}}\right), \quad \ldots \quad (59)$$

the initial conditions being as in (57) with the appropriate Y_0.

The case $\alpha = \frac{1}{4}$ can be similarly treated.

22. *The Case of Quasi-constant Burning Surface Shapes.*

There is an important class of shapes for which θ, in the form function, is small, so that the burning surface remains very nearly constant, and so

$$F = \int_0^{\phi} \frac{d\phi}{\phi^{\alpha}\sqrt{(1+\theta)^2 - 4\theta\phi}} = \frac{1}{1+\theta}\left\{\frac{\phi^{1-\alpha}}{1-\alpha} + \tfrac{1}{2}\varepsilon \cdot \frac{\phi^{2-\alpha}}{2-\alpha} + \frac{1\cdot 3}{2\cdot 4} \cdot \varepsilon^2 \cdot \frac{\phi^{3-\alpha}}{3-\alpha} + \ldots\right\},$$

where $\varepsilon = 4\theta/(1+\theta)^2$ and is small.

For values of θ equal to $0 \cdot 1$ or less it is possible to neglect terms involving powers of ε beyond the first, so that, approximately,

$$(1 + \theta)\,\mathrm{F} = \frac{\phi^{1-\alpha}}{1-\alpha} + \tfrac{1}{2}\varepsilon\,\frac{\phi^{2-\alpha}}{2-\alpha}.$$

(Taking $\varepsilon = 0 \cdot 3$, $\alpha = \frac{1}{2}$, and the maximum value of ϕ, viz., unity, the terms in the bracket for F above are 2, $0 \cdot 1$, $0 \cdot 0135$, etc., so that the approximation is justifiable.)

We easily get the following expression for ϕ, retaining the first power of ε only :

$$\phi = \mathrm{X}^{\frac{1}{1-\alpha}}\left[1 - \frac{1}{2-\alpha}\cdot\frac{2\theta}{(1+\theta)^2}\,\mathrm{X}^{\frac{1}{1-\alpha}}\right], \quad \ldots \ldots \ldots \quad (60)$$

where $\mathrm{X} = (1-\alpha)(1+\theta)\,\mathrm{F}$ and ε has been replaced by its value in terms of θ.

The fundamental equation (22) is now

$$\mathrm{Y}\frac{d^2\mathrm{Y}}{d\mathrm{X}^2} = \mathrm{X}^{\frac{1}{1-\alpha}}\left[1 - \frac{2\theta}{(2-\alpha)(1+\theta)^2}\,\mathrm{X}^{\frac{1}{1-\alpha}}\right], \quad \ldots \ldots \ldots \quad (61)$$

where

$$\left.\begin{aligned} \mathrm{X} &= (1-\alpha)(1+\theta)\,\mathrm{F} = \phi^{1-\alpha} \\ \mathrm{Y} &= \frac{(1+\theta)(1-\alpha)^{\frac{1}{2}}\,\lambda^{\alpha-\frac{1}{2}}\mu^{\frac{1}{2}}\beta}{\mathrm{D}}\,y \end{aligned}\right\}, \quad \ldots \ldots \ldots \ldots \quad (62)$$

the initial conditions being of the same type as above.

25. *Methods of Solution* ($\alpha = \frac{1}{2}, \frac{3}{4}$).

Equations of the type

$$\mathrm{Y}\frac{d^2\mathrm{Y}}{d\mathrm{X}^2} = f(\mathrm{X}), \quad \text{with } \mathrm{X} = 0, \quad \mathrm{Y} = \mathrm{Y}_0, \quad \frac{d\mathrm{Y}}{d\mathrm{X}} = 0,$$

yield the solution in series

$$\mathrm{Y} = \mathrm{Y}_0 + \frac{f(0)}{\mathrm{Y}_0}\cdot\frac{\mathrm{X}^2}{2!} + \frac{f'(0)}{\mathrm{Y}_0}\cdot\frac{\mathrm{X}^3}{3!} + \left\{\frac{f''(0)}{\mathrm{Y}_0} - \frac{f(0)}{\mathrm{Y}_0^3}\right\}\frac{\mathrm{X}^4}{4!} + \ldots,$$

provided that the successive differential coefficients of $f(\mathrm{X})$ exist and are finite for $\mathrm{X} = 0$. Further, only odd inverse powers of Y_0 occur, so that the solution can be written as

$$\mathrm{Y} = \mathrm{Y}_0 + \mathrm{L}(\mathrm{X})\frac{1}{\mathrm{Y}_0} + \mathrm{M}(\mathrm{X})\frac{1}{\mathrm{Y}_0^3} + \mathrm{N}(\mathrm{X})\frac{1}{\mathrm{Y}_0^5} + \ldots, \quad \ldots \ldots \quad (63)$$

where L (X), M (X), N (X), etc., are functions of X which, if obtained as above, would be given as infinite power series.

Assuming, however, a solution as in (63), and substituting in the differential equation, we have

$$\left(\mathrm{Y}_0 + \frac{\mathrm{L}}{\mathrm{Y}_0} + \frac{\mathrm{M}}{\mathrm{Y}_0^3} + \dots\right)\left(\frac{\mathrm{L}_2}{\mathrm{Y}_0} + \frac{\mathrm{M}_2}{\mathrm{Y}_0^3} + \dots\right) = f(\mathrm{X}), \quad \dots \dots \quad (64)$$

suffixes denoting differentiation.

To solve the equation, (64) must be satisfied identically together with the initial conditions $\mathrm{X} = 0$, $\mathrm{L} = \mathrm{L}_1 = 0$, etc., etc.

There is thus obtained a series of *linear* differential equations as below:

$$\left.\begin{aligned} \mathrm{L}_2 &= f(\mathrm{X}) \\ \mathrm{M}_2 + \mathrm{LL}_2 &= 0 \\ \mathrm{N}_2 + \mathrm{LM}_2 + \mathrm{L}_2\mathrm{M} &= 0, \text{etc.}, \end{aligned}\right\} \quad \dots \dots \quad (65)$$

which can be solved in succession by straightforward integration. Equations (55) and (59) can obviously be dealt with by this method, although the auxiliary functions become very complicated.

Thus, for $f(\mathrm{X}) = \sin^2 \mathrm{X}$, we find

$$\mathrm{L} = \tfrac{1}{4}(\mathrm{X}^2 - \sin^2 \mathrm{X})$$

$$\mathrm{M} = -\frac{39}{512} + \frac{3\mathrm{X}^2}{64} - \frac{\mathrm{X}^4}{96} + \frac{5}{64}\cos 2\mathrm{X} - \frac{1}{512}\cos 4\mathrm{X} + \frac{\mathrm{X}}{16}\sin 2\mathrm{X} - \frac{\mathrm{X}^2}{32}\cos 2\mathrm{X}.$$

Theoretically, the complete solution could be so constructed, but the process of numerical integration is more practicable.

From (54) and (56), $\sin^2 \mathrm{X} = 4\theta\phi/(1+\theta)^2$, so that at burnt, $\sin^2 \mathrm{X} = 4\theta/(1+\theta)^2$, which for the cord form ($\theta = 1$) gives $\mathrm{X} = \pi/2$.

The corresponding values of L and M for $\mathrm{X} = \pi/2$ are $0\cdot36685$ and $0\cdot02579$, so that the successive functions may be expected to diminish rapidly in importance. The solution given by (63) may be expressed symbolically as

$$\mathrm{Y} = \mathrm{Y}_0 \,\mathfrak{L}(\mathrm{X}) \quad \dots \dots \quad (66)$$

where $\mathfrak{L}(\mathrm{X})$ is a function which depends also on Y_0, this marking the difference from the case considered in Art. 14, and we cannot therefore obtain a monomial relation for maximum pressure corresponding to (43).

24. *The Pressure Curve.*

We have from (11)

$$\text{for } \alpha = \tfrac{1}{2},\ p = \frac{\lambda\phi(f)}{x} = \frac{\lambda\phi(f)}{y^2}$$

and so by (54) and (56),

$$p = \frac{2\lambda\mu\beta^2\theta}{\mathrm{D}^2}\frac{\sin^2\mathrm{X}}{\mathrm{Y}^2}. \quad \dots \dots \quad (67)$$

The equation to give X at maximum pressure is, therefore,

$$Y - \tan X \frac{dY}{dX} = 0,$$

and by (66) the solution, X_m say, is a function of Y_0, and so is $\sin^2 X_m/Y_m^2$. Hence, from (67), $p_m D^2/\lambda \mu \beta^2 \theta$ may be considered as a function of

$$Y_0^2 = \frac{8\mu\theta^2\beta^2 l}{D^2(1+\theta)^2} \qquad \text{(equation (56))}.$$

It follows that for a propellant of given burning characteristics ($\alpha = \frac{1}{2}$, β known), $p_m D^2\sigma^2/\varpi w_1 \theta$ may be plotted against $w_1 E \theta^2/\sigma^2 D^2 (1+\theta)^2$ (since $\lambda \propto \varpi/\sigma$, $\mu \propto w_1/\sigma$, $E = \sigma l$), the resulting curve being called a *pressure curve*, and such a curve can be constructed if a sufficient number of firing results are available.

The maximum pressure corresponding to any other set of conditions for the same propellant (*i.e.*, similar as regards chemical and physical properties) can then be found from the curve by interpolation, or by extrapolation if necessary.

To complete the scheme for ballistic calculation we have the velocity formula

$$v = \frac{2\lambda^{\frac{1}{2}}(1+\theta)}{\sqrt{8\mu\theta}} \frac{dY}{dX}, \qquad \ldots \ldots \ldots \ldots \qquad (68)$$

and, at burnt, $\sin X = 2\theta^{\frac{1}{2}}/(1+\theta)$.

Similar results can be obtained for $\alpha = \frac{3}{4}$ using (59).

25. *The Method of Solution for the Case of Art.* 22.

On attempting to solve (61) by a series of the type (36) it is easily found that $m_1 = \gamma + 2$, $m_2 = 2\gamma + 2$, $m_3 = 2\gamma + 4$, etc., etc., where $\gamma = 1/(1-\alpha)$, so that we assume as a solution

$$Y = Y_0 + a_1 X^{\gamma+2} + a_2 X^{2\gamma+2} + a_3 X^{2\gamma+4} + a_4 X^{3\gamma+4} + \quad \ldots \ldots \quad (69)$$

the powers of X increasing by γ and 2 alternately. Determining the coefficients, it is found that $a_1, a_3, a_5 \ldots$ are multiples of $1/Y_0$, $1/Y_0^3$, $1/Y_0^5 \ldots$, and $a_2, a_4, a_6 \ldots$ of κ/Y_0, κ/Y_0^3, $\kappa/Y_0^5 \ldots$ ($\kappa = 2\theta/(2-\alpha)(1+\theta)^2$), the multipliers involving γ only.

There are, however, a number of terms in the product $Y d^2Y/dX^2$ which are uncompensated, but they involve the square of the small quantity κ, and so may be neglected.

An approximate solution of (61) is, therefore, of the form

$$\frac{Y}{Y_0} = \left\{1 + \alpha_1\left(\frac{X^{\gamma+2}}{Y_0^2}\right) + \alpha_2\left(\frac{X^{\gamma+2}}{Y_0^2}\right)^2 + \alpha_3\left(\frac{X^{\gamma+2}}{Y_0^2}\right)^3 + \ldots\right\}$$
$$+ \kappa X^{\gamma}\left\{\alpha_1'\left(\frac{X^{\gamma+2}}{Y_0^2}\right) + \alpha_2'\left(\frac{X^{\gamma+2}}{Y_0^2}\right)^2 + \alpha_3'\left(\frac{X^{\gamma+2}}{Y_0^2}\right)^3 + \ldots\right\} \quad (70)$$

where α_1, α_1', etc., involve γ, *i.e.*, α alone, the coefficients α_1, α_2 ... being, of course, the same as A_1, A_2 ... in (38), and so we can write

$$\frac{Y}{Y_0} = \mathcal{L}(z) + \kappa X^{\gamma} M(z) \quad \ldots \ldots \ldots \ldots \quad (71)$$

where $\mathcal{L}(z)$ is the function of Art. 14, and M (z) is a new function. From (11) and (60) we get $p = \lambda X^{\gamma}(1 - \kappa X^{\gamma})/y^{\gamma}$, *i.e.*, p varies as $X^{\gamma}(1 - \kappa X^{\gamma})/Y^{\gamma}$, so that, at maximum pressure,

$$Y(1 - 2\kappa X^{\gamma}) - X(1 - \kappa X^{\gamma})\frac{dY}{dX} = 0. \quad \ldots \ldots \ldots \quad (72)$$

The solution for the case $\theta = 0$, *i.e.*, $\kappa = 0$, is z_m defined by (42), and, for small values of κ, the z solution of (72), z_m', say, will not differ much from z_m. Writing $z_m' = z_m(1 + \varepsilon)$, where ε is small, a first approximation to the true value of the maximum pressure can be determined. The following result is obtained, second order powers and products of ε and κ being neglected :—

$$p_m = \frac{\{(1-\alpha)\, z_m\}^{\frac{1}{3-2\alpha}}}{\{\mathcal{L}(z_m)\}^{\frac{1}{1-\alpha}}} \left\{\frac{(1+\theta)^2 \lambda^2 \mu \beta^2}{D^2 l}\right\}^{\frac{1}{3-2\alpha}} \left[1 - \left\{1 + \frac{M(z_m)}{(1-\alpha)\,\mathcal{L}(z_m)}\right\} \kappa X_m^{\frac{1}{1-\alpha}}\right] \quad (73)$$

where $X_m = (z_m Y_0^2)^{\frac{1-\alpha}{3-2\alpha}}$, and Y_0 is given by (62) with $y = l^{1-\alpha}$. With $\theta = 0$ we get, of course, the result given in (43).

The formula for the velocity is

$$v = \frac{1}{(1-\alpha)^{\frac{1}{2}}} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}} \frac{dY}{dX} \quad \ldots \ldots \ldots \ldots \quad (74)$$

The value of X at burnt can be found from (60) by putting $\phi = 1$, and we find, approximately, $X = 1 + (1 - \alpha)\,\kappa$.

When the functions $\mathcal{L}(z)$, M (z) are tabulated ballistic calculations are possible, using (73) and (74).

A note may be inserted here relative to the method of allowing for the effect of band and frictional resistance by assuming a shot-start pressure (*cf.* CHARBONNIER, (2)). This amounts to saying that a small portion of the charge $\phi(f_0) = \phi_0$, say, is burnt before the shot begins to move, so that the true initial value of X is not zero, but is given by $\phi_0 = X_0^{\gamma}(1 - \kappa X_0^{\gamma})$ (equation (60)).

Approximately, since $\kappa\phi_0$ is very small, we have $X_0^{\gamma} = (1 - \sqrt{1 - 4\kappa\phi_0})/2\kappa = \phi_0$, or $X_0 = \phi_0^{1-\alpha}$. For values of $\alpha < 1$, $\phi_0^{1-\alpha}$ is small, but as α approaches unity the value of $\phi_0^{1-\alpha}$ rapidly becomes appreciable.

For example, taking $\phi_0 = \frac{1}{50}$, the values of $\phi_0^{1-\alpha}$ for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ are, respectively, 0·0532, 0·1415, 0·3764, 1. These figures show that allowance for a shot start has comparatively little effect on ballistic calculations for small values of α, and this has been confirmed in many instances.* It is clear, however, that band and frictional resistance must considerably affect ballistics, whatever be the nature of the propellant, so that the shot start method of correction is open to serious objection.

26. *The General Case* $\alpha = p/q$, p, q *Positive Integers, and* $p < q$.

The special cases dealt with above have illustrated the mathematical difficulties of the ballistic problem for shapes admitting the general form function, and we conclude this part of the paper with the discussion of a case sufficiently general for most purposes. (To avoid confusion, P will now denote the pressure.)

Since ψ (F) cannot be determined explicitly, the equation (22) is transformed by taking a new variable ξ, given by $\phi(f) = \xi^q$, so that $\psi(\mathrm{F}) = \xi^q$ and $d\mathrm{F}/df = -1/\phi^\alpha = -\xi^{-p}$.

The new equation is

$$\{(1+\theta)^2 - 4\theta\xi^q\}\, y\frac{d^2y}{d\xi^2} + \left\{\frac{(1+\theta)^2(p-q+1)}{\xi} - 2\theta(2p-q+2)\,\xi^{q-1}\right\} y\frac{dy}{d\xi} = kq^2\xi^{3q-2p-2},$$

and the further transformations

$$\left.\begin{aligned} \mathrm{X} &= \left\{\frac{4\theta}{(1+\theta)^2}\right\}^{\frac{1}{q}}\xi \\ \mathrm{Y} &= \frac{(4\theta)^{\frac{3q-2p}{2q}}}{(1+\theta)^{2\left(1-\frac{p}{q}\right)}}\cdot\frac{y}{qk^{\frac{1}{2}}} \end{aligned}\right\}, \quad \ldots \ldots \ldots \ldots \quad (75)$$

lead to the numerical form

$$(1-\mathrm{X}^q)\,\mathrm{Y}\frac{d^2\mathrm{Y}}{d\mathrm{X}^2} + \left\{\frac{p-q+1}{\mathrm{X}} - \left(p-\frac{q}{2}+1\right)\mathrm{X}^{q-1}\right\}\mathrm{Y}\frac{d\mathrm{Y}}{d\mathrm{X}} = \mathrm{X}^{3q-2p-2} \ldots \quad (76)$$

The initial conditions are, as formerly, $\mathrm{X} = 0$, $\mathrm{Y} = \mathrm{Y}_0$, $d\mathrm{Y}/d\mathrm{X} = 0$, so that we assume as a formal solution $\mathrm{Y} = \mathrm{Y}_0 + \sum_2^\infty a_n\mathrm{X}^n$.

Substituting in (76) the following identical relation is obtained

$$\left[\sum_2^\infty n(n-q+p)\,a_n\mathrm{X}^{n-1} - \sum_2^\infty n\left(n+p-\frac{q}{2}\right)a_n\mathrm{X}^{n+q-1}\right]\left[\mathrm{Y}_0 + \sum_2^\infty a_n\mathrm{X}^n\right] = \mathrm{X}^{3q-2p-1}. \quad (77)$$

It is clear that $a_2 = a_3 = \ldots = a_{3q-2p-1} = 0$, and putting $m = 3q-2p$, $(m > q)$, (77) becomes

$$[(\mathrm{A}_m\mathrm{X}^{m-1} + \mathrm{A}_{m+1}\mathrm{X}^m + \ldots) - (\mathrm{A}'_m\mathrm{X}^{m+q-1} + \mathrm{A}'_{m+1}\mathrm{X}^{m+q} + \ldots)][\mathrm{Y}_0 + a_m\mathrm{X}^m + a_{m+1}\mathrm{X}^{m+1} + \ldots] = \mathrm{X}^{m-1},$$

where A_m, A'_m, etc., are numerical multiples of a_m, etc. From this identity it follows

* See also GOSSOT and LIOUVILLE (3), vol. 2, p. 46.

that $A_{m+1} = A_{m+2} = \ldots = A_{m+q-1} = 0$, and similarly for the accented coefficients, so that the next non-vanishing coefficient is a_{m+q}: thus the above relation may be further modified as below:

$$[(A_m X^{m-1} + A_{m+q} X^{m+q-1} + \ldots) - (A'_m X^{m+q-1} + A'_{m+q} X^{m+2q-1} + \ldots)][Y_0 + a_m X^m + a_{m+q} X^{m+q} + \ldots] = X^{m-1}.$$

From this, since $2m - 1 > m + q - 1$, it is clear that

$$a_{m+q+1} = a_{m+q+2} = \ldots = a_{2m-1} = 0,$$

and also, equating to zero the coefficient of X^{2m-1}, we have

$$(A_{2m} - A'_{2m-q})\, a_0 + A_m a_m = 0,$$

i.e., a_{2m} exists because A_{2m} is not zero.

Now any term in the first bracket which is uncompensated by terms obtained on multiplying up the two series can be written

$$(A_r - A'_{r-q})\, X^{r-1},$$

and, since this must vanish, if $r > m$, a_r exists if a_{r-q} exists (rule 1).

Multiplying up, the general term is

$$[a_0 (A_{r+s} - A'_{r+s-q}) + a_s (A_r - A'_{r-q}) + a_r (A_s - A'_{s-q})]\, x^{r+s-1},$$

and this must vanish, therefore a_{r+s-q} exists if a_r, a_s exist (rule 2). It is known to begin with that a_m, a_{m+q} exist, hence, applying the above rules, the following non-zero coefficients are found

$$a_m,\ a_{m+q},\ a_{m+2q},\ a_{m+3q} \ldots;\quad a_{2m},\ a_{2m+q},\ a_{2m+2q} \ldots;\quad a_{3m},\ a_{3m+q} \ldots, \text{ etc., etc.}$$

As an example, taking $p = 3$, $q = 5$, so that $m = 9$, the series solution of (76) is of the form

$$Y = Y_0 + a_9 X^9 + a_{14} X^{14} + a_{18} X^{18} + a_{19} X^{19} + a_{23} X^{23} + a_{24} X^{24} + \ldots.$$

As regards the coefficients it can be established that a_m, a_{m+q}, $a_{m+2q} \ldots$ are all multiples of $1/Y_0$; a_{2m}, $a_{2m+q} \ldots$ of $1/Y_0^3$; a_{3m}, $a_{3m+q} \ldots$ of $1/Y_0^5$, and so on, so that the solution of (76) is of the form

$$Y = Y_0 + \left(\frac{1}{Y_0}\right)(\alpha_m X^m + \alpha_{m+q} X^{m+q} + \alpha_{m+2q} X^{m+2q} + \ldots) + \left(\frac{1}{Y_0^3}\right)(\alpha_{2m} X^{2m} + \alpha_{2m+q} X^{2m+q} + \ldots) + \left(\frac{1}{Y_0^5}\right)(\alpha_{3m} X^{3m} + \alpha_{3m+q} X^{3m+q} + \ldots), \ldots\ldots, \qquad (78)$$

the α's being numbers depending on p and q.

The law of the coefficients α_m, α_{m+q}, $\alpha_{m+2q} \ldots$ is simple, and can, in fact, be obtained from the identical relation (77) by leaving out the series $\sum_2^\infty a_n X^n$ in the second bracket on the left-hand side.

We find

$$\alpha_m = \frac{1}{m(m-q+p)},$$

$$\alpha_{m+(n+1)q} = \frac{(m+nq)\{m+p+(n-\frac{1}{2})q\}}{\{m+(n+1)q\}\{m+p+nq\}}\alpha_{m+nq} \qquad (n = 0, 1, 2 \ldots)$$

Since product terms occur in the equations determining the other sets of coefficients, simple expressions for them cannot be obtained, but it can be shown that α_{m+nq} ($n = 0$, $1, 2 \ldots$) are all positive, α_{2m+nq} are all negative, α_{3m+nq} all positive, and so alternately, and that they diminish progressively in value.

The method of series suffices to calculate Y in terms of X, provided X is not too close to unity, but in practice a difficulty arises, since dY/dX becomes infinite for $X = 1$, as is shown by the fact that the series obtained by differentiating $\sum_{n=0}^\infty \alpha_{m+nq} X^{m+nq}$ diverges for $X = 1$.

Since $X = 1$ at burnt for $\theta = 1$ (equation (75)), and dY/dX occurs in the expression for the shot velocity, this method is not suitable for ballistic calculations with propellants in the shape of circular cords. Using (75), the pressure and shot velocity in terms of X and Y are

$$\left.\begin{aligned} P &= \lambda\left(\frac{4\theta}{kq^2}\right)^{\frac{q}{2(q-p)}} \frac{X^q}{Y^{\frac{q}{q-p}}} \\ v &= \left(\frac{q}{q-p}\right)^{\frac{1}{2}} \left\{\frac{(1+\theta)^2}{4\theta}\right\}^{\frac{1}{2}} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}} X^{p-q+1}(1-X^q)^{\frac{1}{2}} \frac{dY}{dX} \end{aligned}\right\}, \quad \ldots \quad (79)$$

so that, for $\theta = 1$, the value of v at burnt is of the indeterminate form $0 \times \infty$. For calculation purposes it would be necessary to construct tables of double entry giving Y and dY/dX in terms of Y_0 and X, and for values of X ranging from 0 to 1, one set of tables for each value of α. The range of the parameter Y_0 is difficult to assign beforehand, but it must be sufficient to include all possible ballistic combinations of gun and charge.

X and Y at maximum pressure, X_m, Y_m, say, are calculated from the equation (*cf.* (79))

$$Y = \frac{X}{q-p}\frac{dY}{dX}, \quad \ldots \quad (80)$$

so that X_m, Y_m, apart from p and q, are functions of Y_0.

The value of Y_0 is found from (75), putting $y = x^{1-\alpha} = l^{1-\frac{p}{q}}$ initially, and substituting for k from (21), and we thus find from (79), omitting numerical constants and replacing λ, μ, l in terms of ϖ, σ, E, w_1 that

$$P_m\left(\frac{D^2\sigma^2}{\theta\varpi w_1}\right)^{\frac{1}{2(1-\alpha)}} \text{ is a function of } \frac{\theta^{3-2\alpha}}{(1+\theta)^{4(1-\alpha)}} \cdot \frac{\varpi^{2\alpha-1}E^{2(1-\alpha)}w_1}{D^2\sigma^2}.$$

This is the pressure-curve relation, and, with $\alpha = 1/2$, we get the result of Art. 24.

The Construction of the Ballistic Tables.

27. *The Methods Employed.*

The construction of series solutions of the ballistic equations has been explained in the preceding section, and these can be employed for calculation with small values of the independent variable.

To extend the calculations to make the tables sufficiently comprehensive for general ballistic purposes, it is necessary to employ numerical methods of integration, of which several are available: the one used in computing the tables is that due to Runge.*

It is necessary to start from a set of values of X, Y, dY/dX, and to calculate Y and dY/dX corresponding to values of X increasing by steps (h) of equal value. The closeness of the approximation depends, of course, on the magnitude of h, and it is necessary to examine the work carefully at various stages, to determine the degree of accuracy of the results obtained. A check for small values of X is afforded by a comparison with the results given by direct calculation from the series solutions.

28. *The Case of the Constant Burning Surface Shape.*

The appropriate ballistic function $\mathcal{L}(z)$ has been defined in Art. 14 (equation (40)), and it is found to satisfy the differential equation

$$(2-\alpha)(3-2\alpha)\,\mathcal{L}\frac{d\mathcal{L}}{dz}+(3-2\alpha)^2\,\mathcal{L}z\frac{d^2\mathcal{L}}{dz^2}=(1-\alpha)^2. \quad (81)$$

It is convenient to replace z by $(2-\alpha)(3-2\alpha)\,Z/(1-\alpha)^2$, and then (81) becomes

$$\mathcal{L}\frac{d\mathcal{L}}{dZ}+\frac{3-2\alpha}{2-\alpha}\,\mathcal{L}Z\frac{d^2\mathcal{L}}{dZ^2}=1, \quad (82)$$

the series solution being of the form

$$\mathcal{L}(Z)=1+Z+a_2Z^2+a_3Z^3+\dots, \quad (83)$$

so that the initial conditions are

$$Z=0, \qquad \mathcal{L}(Z)=\mathcal{L}'(Z)=1. \quad (84)$$

The equation to determine Z_m, the value of Z at maximum pressure, is unaltered, and is thus the same as (42).

For any value of α, the functions $\mathcal{L}(Z)$, $\mathcal{L}'(Z)$ can be calculated by the Runge process, starting from the values given by (84), or, alternatively, the calculations may be made using the series solutions for as high a value of Z as possible, and then continued by the numerical method.

* An account of the method for first order equations is given by Forsyth, 'Differential Equations' (pp. 51–54), and a fuller treatment is to be found in Runge, 'Graphical Methods' (Columbia Univ. Press).

There is a slight difficulty to begin with, because the initial value of $d^2\mathcal{L}/dZ^2$ is apparently indeterminate, but we have

$$\underset{Z\to 0}{\mathrm{Lt}}\ \frac{3-2\alpha}{2-\alpha}\frac{d^2\mathcal{L}}{dZ^2}=\underset{Z\to 0}{\mathrm{Lt}}\ \frac{1-\mathcal{L}\dfrac{d\mathcal{L}}{dZ}}{\mathcal{L}Z}=\underset{Z\to 0}{\mathrm{Lt}}\ \frac{\dfrac{d}{dZ}\left(1-\mathcal{L}\dfrac{d\mathcal{L}}{dZ}\right)}{\dfrac{d}{dZ}(\mathcal{L}Z)},$$

and from this we find

$$\left[\frac{d^2\mathcal{L}}{dZ^2}\right]_{Z=0}=-\frac{2-\alpha}{5-3\alpha}.$$

Changing from z to Z the formulæ for pressure and shot velocity are, from (43) and (45),

$$\left.\begin{aligned}
p &= \frac{\left\{\dfrac{(2-\alpha)(3-2\alpha)}{1-\alpha}\,Z\right\}^{\frac{1}{3-2\alpha}}}{\{\mathcal{L}(Z)\}^{\frac{1}{1-\alpha}}}\cdot\left(\frac{\lambda^2\mu\beta^2}{D^2 l}\right)^{\frac{1}{3-2\alpha}}\\
v &= \frac{(3-2\alpha)^{\frac{2-\alpha}{3-2\alpha}}}{(1-\alpha)^{\frac{5-2\alpha}{2(3-2\alpha)}}(2-\alpha)^{\frac{1-\alpha}{3-2\alpha}}}\cdot\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}}(Y_0)^{\frac{1}{3-2\alpha}}(Z)^{\frac{2-\alpha}{3-2\alpha}}\mathcal{L}'(Z)\\
\text{with}\quad \frac{1}{Y_0^{\,2}} &= \frac{D^2}{(1-\alpha)\,l^{2-2\alpha}\,\lambda^{2\alpha-1}\,\mu\beta^2}
\end{aligned}\right\}\quad\ldots\quad(85)$$

and, for maximum pressure,

$$p_m=\frac{\left\{\dfrac{(2-\alpha)(3-2\alpha)}{1-\alpha}\,Z_m\right\}^{\frac{1}{3-2\alpha}}}{\{\mathcal{L}(Z_m)\}^{\frac{1}{1-\alpha}}}\cdot\left(\frac{\lambda^2\mu\beta^2}{D^2 l}\right)^{\frac{1}{3-2\alpha}}=G(\alpha)\left(\frac{\lambda^2\mu\beta^2}{D^2 l}\right)^{\frac{1}{3-2\alpha}}\quad\ldots\quad(86)$$

At burnt,

$$\left.\begin{aligned}
Z &= \frac{(1-\alpha)^2}{(2-\alpha)(3-2\alpha)}\,\frac{1}{Y_0^{\,2}},\\
\text{so that}\quad v &= \frac{(1-\alpha)^{\frac{1}{2}}}{2-\alpha}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}}\frac{1}{Y_0}\cdot\mathcal{L}'(Z)
\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(87)$$

The shot travel in terms of Z is given by

$$\frac{x}{l}=\left(\frac{Y}{Y_0}\right)^{\frac{1}{1-\alpha}}=\{\mathcal{L}(Z)\}^{\frac{1}{1-\alpha}}\quad\ldots\ldots\ldots\ldots\quad(88)$$

Equations (85) to (87), with the help of tables constructed on the plan indicated below, give all the ballistic information required.

It may be noticed that, with $\gamma=1$, (43) and (52) agree, as, of course, should be the case, for the equation connecting ζ and χ is then (*cf.* (46))

$$\zeta\chi\frac{d^2\chi}{d\zeta^2}+\frac{2-\alpha}{3-2\alpha}\chi\frac{d\chi}{d\zeta}=1,$$

and, putting $\zeta=(1-\alpha)^2 z/(3-2\alpha)^2$, this becomes the same as (81) with $\chi=\mathcal{L}(z)$.

29. *The Ballistic Tables.*

These are compiled in the form of six columns giving the corresponding values of

$$\mathrm{Z}; \qquad \mathcal{L}(\mathrm{Z}); \qquad \mathcal{L}'(\mathrm{Z}); \qquad \frac{\left\{\frac{(2-\alpha)(3-2\alpha)}{1-\alpha}\mathrm{Z}\right\}^{\frac{1}{3-2\alpha}}}{\{\mathcal{L}(\mathrm{Z})\}^{\frac{1}{1-\alpha}}}; \qquad \{\mathcal{L}(\mathrm{Z})\}^{\frac{1}{1-\alpha}};$$

$$\frac{(3-2\alpha)^{\frac{2-\alpha}{3-2\alpha}}}{(1-\alpha)^{\frac{5-2\alpha}{2(3-2\alpha)}}(2-\alpha)^{\frac{1-\alpha}{3-2\alpha}}}(\mathrm{Z})^{\frac{2-\alpha}{3-2\alpha}}\mathcal{L}'(\mathrm{Z});$$

together with the values of Z_m, $\mathcal{L}(\mathrm{Z}_m)$ and $\mathrm{G}(\alpha)$.

The range of Z is from zero upwards by steps of 0·1, and the final value of Z to be considered is, of course, the value at burnt (equation (87)). This cannot be assigned beforehand, but the following considerations help to fix it approximately, consistent with cases which are likely to arise in practice.

In (88) x is the shot travel plus l, so that

$$\frac{x}{l} = \frac{\text{travel} + l}{l} = \frac{\mathrm{E} + \text{bore capacity}}{\mathrm{E}} \text{ since } \mathrm{E} = \sigma l.$$

Hence, at burnt,

$$\frac{x}{l} = \frac{\text{total capacity to burnt} - \text{space occupied by charge}}{\text{chamber capacity} - \text{space occupied by charge}}.$$

so that

$$\frac{x}{l} > \frac{\text{total capacity to burnt}}{\text{chamber capacity}}.$$

Thus, since burnt may occur at the muzzle in extreme cases, a reasonable outside value of x/l is the ratio of the total capacity of the gun to chamber capacity, which quantity varies considerably from gun to gun. To include guns of small chamber capacity and high densities of loading, a limiting value of x/l of about 10 is necessary, and will probably cover all cases of practical importance.

Therefore, by (88), the limiting value of $\mathcal{L}(\mathrm{Z})$ is to be about $(10)^{1-\alpha}$, which for $\alpha = \frac{1}{2}$ is 3·162, for $\alpha = \frac{2}{3}$, 2·155, and so on.

Keeping the steps in Z equal to 0·1, the tables for low values of α are rather lengthy, but the calculations offer no inherent difficulty. A series of such tables for $\alpha = \frac{1}{2}$ and upwards has been computed, and a typical specimen ($\alpha = \frac{1}{2}$) is reproduced below.

It should be noticed that the case $\alpha = 0$, corresponding to a rate of burning *independent* of the pressure, falls into the above scheme, although the appropriate ballistic tables would be very lengthy.

TABLE I.—$\alpha = \frac{1}{2}$.

Col. 1.	Col. 2.	Col. 3.	Col. 4.	Col. 5.	Col. 6.
Z	$\mathcal{L}(Z)$	$\mathcal{L}'(Z)$	$\dfrac{(6Z)^{\frac{1}{2}}}{\{\mathcal{L}(Z)\}^2}$	$\{\mathcal{L}(Z)\}^2$	$\dfrac{4}{3^{\frac{1}{4}}} \cdot Z^{\frac{3}{4}} \mathcal{L}'(Z)$
0	1·0	1·0	0·0	1·0	0·0
0·1	1·09796	0·96018	0·64254	1·20552	0·51896
0·2	1·19221	0·92548	0·77070	1·42136	0·84124
0·3	1·28319	0·89485	0·81481	1·64658	1·10248
0·4	1·37128	0·86749	0·82386	1·88041	1·32614
0·5	1·45678	0·84284	0·81615	2·12221	1·52318
0·6	1·53993	0·82045	0·80011	2·37138	1·69998
0·7	1·62093	0·79997	0·78000	2·62741	1·86070
0·8	1·69996	0·78114	0·75813	2·88986	2·00828
0·9	1·77721	0·76374	0·73573	3·15848	2·14490
1·0	1·85276	0·74758	0·71357	3·43272	2·27215
1·1	1·92676	0·73252	0·69200	3·71240	2·39135
1·2	1·99930	0·71842	0·67129	3·99720	2·50348
1·3	2·07047	0·70510	0·65149	4·28685	2·60911
1·4	2·14035	0·69266	0·63265	4·58114	2·70954
1·5	2·20903	0·68092	0·61478	4·87981	2·80507
1·6	2·27656	0·66987	0·59783	5·18273	2·89641
1·7	2·34302	0·65934	0·58177	5·48974	2·98350
1·8	2·40845	0·64934	0·56655	5·80063	3·06694
1·9	2·47290	0·63981	0·55213	6·11524	3·14700
2·0	2·53643	0·63074	0·53845	6·43348	3·22406
2·1	2·59906	0·62207	0·52548	6·75512	3·29825
2·2	2·66085	0·61378	0·51315	7·08012	3·36995
2·3	2·72183	0·60583	0·50144	7·40836	3·43896
2·4	2·78202	0·59822	0·49030	7·73963	3·50589
2·5	2·84147	0·59090	0·47969	8·07395	3·57066
2·6	2·90021	0·58387	0·46957	8·41122	3·63351
2·7	2·95826	0·57711	0·45992	8·75130	3·69455
2·8	3·01564	0·57059	0·45071	9·09409	3·75381
2·9	3·07238	0·56430	0·44190	9·43952	3·81143
3·0	3·12851	0·55824	0·43347	9·78758	3·86760
3·1	3 18404	0·55237	0·42540	10·13811	3·92222

$Z_m = 0 \cdot 39238$; $\mathcal{L}(Z_m) = 1 \cdot 36467$; $G(\tfrac{1}{2}) = 0 \cdot 82390$.

Maximum pressure, $p_m = (0 \cdot 82390)\left(\dfrac{\lambda^2 \mu \beta^2}{D^2 l}\right)^{\frac{1}{2}}$.

Pressure corresponding to prescribed value of Z . $p = \left(\dfrac{\lambda^2 \mu \beta^2}{D^2 l}\right)^{\frac{1}{2}} \times$ Col. 4.

Shot travel . . . . $\dfrac{x}{l} =$ Col. 5.

Shot velocity . . . . $v = \left(\dfrac{\lambda}{\mu}\right)^{\frac{1}{2}} Y_0^{\frac{1}{2}} \times$ Col. 6.

where $\dfrac{1}{Y_0^2} = \dfrac{2D^2}{l \mu \beta^2}$.

Value of Z at burnt, $Z = \dfrac{D^2}{6 l \mu \beta^2}$.

N.B.—If value of Z at burnt is less than Z_m, p_m as given above is the theoretical maximum pressure only, the true maximum being the pressure at burnt.

As a preliminary check, values of $\zeta(Z)$ for the smaller values of Z can be calculated from the series (83). Determining the coefficients, it is found that a_2 is negative, a_3 positive, and so alternately, whilst the coefficients begin to increase in numerical value after a_7.

It is better to take $\zeta = Z/(1+Z)$ as the independent variable, so that $\zeta < 1$: the coefficients then, although all positive, steadily diminish in value. The RUNGE process has the great advantage of determining $\zeta'(Z)$ step by step with $\zeta(Z)$. To get tables for values of α intermediate between two values for which tables have been compiled, it is usually sufficient to use linear interpolation.

Values of $G(\alpha)$, equation (86), have been calculated for a number of cases, and the following table compiled by interpolation from the results.

TABLE II.

α	$G(\alpha)$	α	$G(\alpha)$
0·00	0·99322	0·55	0·79646
0·05	0·97968	0·60	0·76575
0·10	0·96634	0·65	0·73151
0·15	0·95306	0·70	0·69342
0·20	0·93883	0·75	0·65118
0·25	0·92393	0·80	0·60456
0·30	0·90778	0·85	0·55316
0·35	0·89005	0·90	0·49672
0·40	0·87038	0·95	0·43504
0·45	0·84844	1·00	0·36788
0·50	0·82390	—	—

The value of Z_m, defined by (42), has been obtained by interpolation, using the tables; but it may be calculated as follows :—

Modifying (83) to make a_2, a_3, etc., positive, we have

$$\zeta(Z) = 1 + Z - a_2Z^2 + a_3Z^3 - \ldots.$$

so that (42) gives, for the determination of Z_m,

$$0 = 1 - \left(\frac{3-2\alpha}{1-\alpha} - 1\right) Z + \left(2 \cdot \frac{3-2\alpha}{1-\alpha} - 1\right) a_2Z^2 - \left(3 \cdot \frac{3-2\alpha}{1-\alpha} - 1\right) a_3Z^3 + \ldots.$$

Reversing this series, and substituting for a_2, a_3 ... we find

$$Z_m = \frac{1-\alpha}{2-\alpha} + \tfrac{1}{2}\left(\frac{1-\alpha}{2-\alpha}\right)^2 + \tfrac{1}{6}\cdot\left(\frac{1-\alpha}{2-\alpha}\right)^3 \cdot \frac{3-2\alpha}{5-3\alpha} - \tfrac{1}{24}\left(\frac{1-\alpha}{2-\alpha}\right)^4 \cdot \frac{\alpha(3-2\alpha)}{(5-3\alpha)(8-5\alpha)}$$
$$- \tfrac{1}{60}\cdot\left(\frac{1-\alpha}{2-\alpha}\right)^5 \cdot \frac{(3-2\alpha)(72-173\alpha+122\alpha^2-24\alpha^3)}{(5-3\alpha)^2(8-5\alpha)(11-7\alpha)} \quad \ldots \qquad (89)$$

For $\alpha = 1/2$ upwards, (89) gives results in accordance with those obtained from the tables, five-place agreement being found with the value $\alpha = 1/2$.

For $\alpha = 0$, (89) gives $Z_m = 0 \cdot 63745$, which is probably correct to the fourth place, since the last term written down in the series has the value $0 \cdot 0000512$.

$\mathcal{L}(Z_m)$ is fairly easily calculated for such a value of Z_m (and for lesser values), so that the results of Table II can be verified.

30. *The Limiting Velocity.*

From (87) it is seen that the velocity at burnt varies as $Z^{\frac{1}{2}} \mathcal{L}'(Z)$, since the value of Z at burnt is directly proportional to $1/Y_0^2$.

This latter quantity increases with D, and it is interesting to consider $\underset{Z \to \infty}{\mathrm{Lt}} Z^{\frac{1}{2}} \mathcal{L}'(Z)$, which can be taken to apply, in the limit, to the case of a very long gun burning a very large size of propellant.

For a given charge weight the velocity acquired must, of course, be finite, so that $\underset{Z \to \infty}{\mathrm{Lt}} Z^{\frac{1}{2}} \mathcal{L}'(Z)$ must be finite, and the possibility of this can be seen from the figures in Table I.

An upper limit to the value of $Z^{\frac{1}{2}} \mathcal{L}'(Z)$ can be obtained as follows :—

Equation (82) can be written, putting $\gamma = (3 - 2\alpha)/(2 - \alpha)$,

$$\frac{d\mathcal{L}}{dZ} = \frac{Z^{-\frac{1}{\gamma}}}{\gamma} \int_0^Z \frac{dZ}{Z^{1-\frac{1}{\gamma}} \mathcal{L}(Z)} = \frac{1}{\overline{\mathcal{L}}},$$

where $\overline{\mathcal{L}}$ is some value of $\mathcal{L}$ in the range of integration.

Hence

$$\mathcal{L} = 1 + \int_0^Z \frac{dZ}{\overline{\mathcal{L}}} = 1 + \frac{Z}{\overline{\overline{\mathcal{L}}}},$$

where $\overline{\overline{\mathcal{L}}}$ is again some value of $\mathcal{L}$ in the range.

Clearly $1 < \overline{\overline{\mathcal{L}}} < \mathcal{L}$, so we obtain the inequalities

$$\tfrac{1}{2} + \sqrt{Z + \tfrac{1}{4}} < \mathcal{L}(Z) < 1 + Z. \quad \dots \dots \dots \quad (90)$$

It can also be shown that

$$\underset{Z \to \infty}{\mathrm{Lt}} Z^{\frac{1}{2}} \mathcal{L}'(Z) = \frac{1}{\gamma} \underset{Z \to \infty}{\mathrm{Lt}} \frac{1}{Z^{\frac{1}{2}}} \int_0^Z \frac{dZ}{\mathcal{L}(Z)},$$

and so, by using (90), it follows that

$$\underset{Z \to \infty}{\mathrm{Lt}} Z^{\frac{1}{2}} \mathcal{L}'(Z) < \frac{2}{\gamma}.$$

To get closer values is not easy; perhaps the best way is to apply the method of Art. (23) to (82) by assuming

$$\mathcal{L}(\mathrm{Z}) = \mathcal{L}_0 + \gamma\mathcal{L}_1 + \gamma^2\mathcal{L}_2 + \dots,$$

where $\mathcal{L}_0$, $\mathcal{L}_1$, etc., are functions of Z with the conditions

$$\mathrm{Z} = 0, \quad \mathcal{L} = \mathcal{L}_0' = 1, \quad \mathcal{L}_1 = \mathcal{L}_2 = \dots = 0, \quad \mathcal{L}_1' = \mathcal{L}_2' = \dots = 0.$$

It is found that the limits of $\mathrm{Z}^{\frac{1}{2}}\mathcal{L}_0'$, $\mathrm{Z}^{\frac{1}{2}}\mathcal{L}_1'$, etc., when $\mathrm{Z} \rightarrow \infty$, are all finite, and we obtain

$$\underset{\mathrm{Z}\rightarrow\infty}{\mathrm{Lt}}\ \mathrm{Z}^{\frac{1}{2}}\mathcal{L}'(\mathrm{Z}) = \frac{1}{\sqrt{2}}\left(1 + \frac{\gamma}{4} + \frac{3\gamma^2}{32} + \frac{7\gamma^3}{256} + \dots\right). \quad \dots\dots \quad (91)$$

The auxiliary functions become complicated, but the method is direct, and we can, in this way, find the limiting velocity obtainable with a given charge weight in a given chamber, varying the size of the propellant and the length of the gun.

31. *The Case of the General Shape.*

For particular values of α, such as $\alpha = \frac{1}{2}$, the equations of Art. 21 can be integrated by the RUNGE process and tables compiled, checks being made by comparison with the results obtained by the methods of Art. 23: in other cases the method of Art. 26 is available only when $\theta < 1$.

To avoid the difficulty of $d\mathrm{Y}/d\mathrm{X}$ becoming infinite at $\mathrm{X} = 1$, which invalidates the numerical method near $\mathrm{X} = 1$, (76) is transformed by writing

$$\mathrm{X}^q = \sin^2\psi \quad \dots\dots\dots\dots \quad (92)$$

This leads to

$$\mathrm{Y}\frac{d^2\mathrm{Y}}{d\psi^2} + (2\alpha - 1)\cot\psi \,.\, \mathrm{Y}\frac{d\mathrm{Y}}{d\psi} = \frac{4}{q^2}(\sin\psi)^{4-4\alpha}, \quad \dots\dots \quad (93)$$

For $\mathrm{X} = 0$, $\psi = 0$, and $\mathrm{Y} = \mathrm{Y}_0$; also $d\mathrm{Y}/d\psi = \dfrac{\sin 2\psi}{q\mathrm{X}^{q-1}}\dfrac{d\mathrm{Y}}{d\mathrm{X}}$ and from Art. 26 it is known that $d\mathrm{Y}/d\mathrm{X}$ starts with the term $\mathrm{X}^{3q-2p-1}$.

Hence, clearly, since $q > p$, $d\mathrm{Y}/d\psi = 0$, for $\psi = 0$.

Putting $\mathrm{Y} = \mathrm{Y}_0\eta$ in (93)

$$\eta\frac{d^2\eta}{d\psi^2} + (2\alpha - 1)\cot\psi \,.\, \eta\frac{d\eta}{d\psi} = \mathrm{M}(\sin\psi)^{4-4\alpha}, \quad \dots\dots \quad (94)$$

where $\mathrm{M} = 4/q^2\mathrm{Y}_0^2$, the initial conditions being $\psi = 0$, $\eta = 1$, $d\eta/d\psi = 0$.

From (75), (79), and (92) the following expressions are obtained :—

$$\left.\begin{aligned} P &= \frac{\lambda}{l} \cdot \frac{(1+\theta)^2}{4\theta} \cdot \frac{\sin^2\psi}{\eta^{\frac{1}{1-\alpha}}} \\ v &= \left(\frac{1}{1-\alpha}\right)^{\frac{1}{2}} \left\{\frac{(1+\theta)^2}{4\theta}\right\}^{\frac{1}{2}} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}} \frac{(\sin\psi)^{2\alpha-1}}{M^{\frac{1}{2}}} \cdot \frac{d\eta}{d\psi} \\ \eta &= \left(\frac{x}{l}\right)^{1-\alpha} \\ M &= \frac{\left(\frac{1+\theta}{2}\right)^{4-4\alpha}}{\theta^{3-2\alpha}} \cdot \frac{D^2(1-\alpha)}{l^{2-2\alpha}\lambda^{2\alpha-1}\mu\beta^2} \end{aligned}\right\} \quad \ldots\ldots \quad (95)$$

Equation (94) is integrated by the RUNGE method, and the results are tabulated for a range of values of M. For ballistic purposes tables* have been compiled with M ranging from 0·2 to 4·0 by steps of 0·1, and for values of α, 0·5 to 0·9 by 0·1. These give η, $\sin^2\psi/\eta^{1/(1-\alpha)}$, $(\sin\psi)^{2\alpha-1}$, $d\eta/d\psi$ in terms of ψ, so that, by (92), P, v and shot travel are determined for any value of ψ.

At burnt $X^q = \sin^2\psi = 4\theta/(1+\theta)^2$, so that the greatest value of ψ to be considered is $\pi/2$.

The value ψ_m (at maximum pressure) is determined from

$$\frac{d\eta}{d\psi} = 2(1-\alpha)\,\eta\cot\psi \quad \ldots\ldots\ldots\ldots \quad (96)$$

and has therefore to be calculated for each α and each M, the values of $\sin^2\psi_m/\eta_m^{\frac{1}{1-\alpha}}$ being tabulated.

SOME GENERAL CONSIDERATIONS.

32. *Effects of Variations in the Burning Characteristics. Constant Burning Surface Propellants.*

Variations in β correspond to variations in D, so that, ballistically, the slow burning of a propellant, due to a low value of β, may be compensated by a diminution in size, and this is the usual practice.

The effect of variations in α is more difficult to determine, because there are no simple expressions for Z_m and $\mathcal{L}(Z_m)$.

Equation (88) shows that the position of the shot at maximum pressure depends only on α and l—*i.e.*, for a given gun on α and the charge weight, and with the aid of the ballistic tables we find the following results :—

α	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{6}{7}$	1
x_m/l	1·5745	1·8623	2·0439	2·1490	2·2308	2·3394	2·7183

* Not reproduced here from considerations of space.

These figures show that, for a given charge weight, the position of maximum pressure moves steadily forward at an increasing rate as α increases. (A knowledge of the position of maximum pressure is of the first importance in gun design.)

As regards the effect of variations in α on the value of the maximum pressure, the following considerations are of interest.

To determine the burning characteristics of a propellant from closed vessel observations, values of βp^{α} (derived from $\mathrm{D}df/dt$) are plotted against p, and, if a linear rate of burning law is being sought, the best mean straight line is drawn through the plotted points, some of which will lie above and some below the line. We may represent this process by making the values of βp^{α} agree for a definite value p_0, say, so that for a varying α we have a series of curves all passing through the same point.

Putting $\beta p_0{}^{\alpha} = c$, (86) gives

$$p_m = \mathrm{G}(\alpha)\left(\frac{\lambda^2\mu}{\mathrm{D}^2 l}\right)^{\frac{1}{3-2\alpha}}\left(\frac{c}{p_0{}^{\alpha}}\right)^{\frac{2}{3-2\alpha}},$$

so that, for $\alpha = 1$,

$$(p_m)_{\alpha=1} = \frac{1}{e}\cdot\frac{\lambda^2\mu}{\mathrm{D}^2 l}\cdot\left(\frac{c}{p_0}\right)^2.$$

Thus we can write

$$p_m/(p_m)_{\alpha=1} = \mathrm{G}(\alpha)\, e^{\frac{1}{3-2\alpha}}\,(p_0/(p_m)_{\alpha=1})^{\frac{2(1-\alpha)}{3-2\alpha}}.$$

Now in determining α, β from closed vessel firings, the range of pressures must be taken as approximately the same as those expected on firing in the gun, so that p_0 will be $< (p_m)_{\alpha=1}$, but usually not very much less. The ratio $p_0/(p_m)_{\alpha=1}$ can be taken as a constant k (< 1), say, so that $p_m/(p_m)_{\alpha=1} = k\mathrm{G}(\alpha)\,(e/k)^{\frac{1}{3-2\alpha}}$. As α increases from zero, $\mathrm{G}(\alpha)$ decreases, and $(e/k)^{\frac{1}{3-2\alpha}}$ increases, so that there is a possibility of a maximum value for $p_m/(p_m)_{\alpha=1}$ in the range 0 to 1. The following tables illustrate this fact.

$k = \frac{2}{3}$	
α	$p_m/(p_m)_{\alpha=1}$
0·0	1·0578
0·25	1·0807
0·50	1·1091
0·75	1·1080
1·0	1·0000

$k = \frac{1}{2}$	
α	$p_m/(p_m)_{\alpha=1}$
0·0	0·8732
0·25	0·9094
0·50	0·9605
0·75	1·0067
1·0	1·0000

The value of k is much more likely to be near $\frac{2}{3}$ than $\frac{1}{2}$, so that in general the assumption of a linear law of burning leads to an under calculation of maximum pressure.

To determine the effect on the calculated muzzle-velocity of adopting a mean linear rate of burning is impossible without reference to a specific gun, but calculations for a series of guns show that muzzle velocities, corresponding to the linear law, are found to be considerably lower than those determined using an index law. These facts illustrate the great importance of determining the burning law as accurately as possible if correct ballistic predictions are to be made.

As regards propellants of other than constant burning surface shape, little can be done to determine generally the effect on ballistics of variations in α, because of the complexity of the formulæ, but particular calculations show that the above remarks hold good also in this case.

33. *Experimental Determination of Propellant Characteristics.*

The propellant characteristics are f the " force " and α, β, the constants in the rate of burning equation, and it has been the practice in the past to determine these quantities from the results of closed vessel experiments (*cf.* Charbonnier, (2), Chap. II). From the theoretical results obtained in this paper it appears possible that such determinations could be carried out in the gun itself. For, from equation (86), it is seen that the maximum pressure depends, for a fixed α and β, on the value of $\lambda^2\mu/D^2l$, *i.e.*, on $\lambda'^2\mu/D^2l$, where $\lambda = \lambda' f$ and λ' depends only on gun and charge constants.

From the examination of varied firing results with the same (constant burning surface) propellant (*i.e.*, varying charge weight and size), it should, therefore, be possible to deduce a value of α by plotting $\log \lambda'^2\mu/D^2l$ against $\log P_m$, and finding the slope of the resulting mean straight line. Given α, f and β can, theoretically, be determined from the results of two firings, but it will be necessary, of course, to check the values so obtained by a comparison of calculated and measured muzzle velocities, positions of maximum pressure, etc.

Calculations have shown that the differences in ballistics due to variations in α and β are sufficiently marked to permit of the determination of these quantities by careful experiment.

34. *Summary and Conclusions.*

The aim of the paper has been to lay down a foundation for the development of a working system of internal ballistics based on a pressure-index law of burning of propellants. This problem has been attacked in the past, but the analytical difficulties have apparently obstructed progress, and most writers have been content to work out the theory on the simpler assumption of a rate of burning directly proportional to the pressure. The fundamental ballistic equations divide into two main groups, and, by the use of the simpler, the general problem can be solved in a satisfactory manner by the tabulation of certain functions. These ballistic functions arise as the solutions of certain non-linear ordinary differential equations, and methods of solution are described and certain properties of the functions deduced.

In particular, compact monomial relations for the maximum pressure arise in the case of propellants of shapes preserving a constant surface during the burning, and this happens whichever set of ballistic equations be employed. A comparison of the results shows that the modification introduced by the conception of an "expanded" pressure, by means of which the expression for the gas pressure in the case of a varying capacity, as in the gun, takes the same form as in the case of a constant capacity, as in the closed vessel, is justifiable within limits, and the simplification of the resulting analysis makes possible the solution of the general ballistic problem.

The construction of the necessary ballistic tables by a process of numerical integration is described, and methods are given of checking the results by approximate solutions of the equations.

Throughout the paper only the ideal case with no band or frictional resistance has been considered, and the treatment consists mainly of a discussion of the fundamental equations of the subject and their solution.

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